ELEMENTARY TRIGONOMETRY

PARTS I & II

C. V. DURELL & R. M. WRIGHT

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PARTS I—II

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LONDON

G. BELL AND SONS, LTD.

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This book is issued complete and also in three separate parts. Parts I. and II. are available bound together.

LONDON: G. BELL AND SONS, LTD.

Seventeenth Impression
First published 1927

PRINTED IN GREAT BRITAIN BY ROBERT MACLEHOSE AND CO. LTD.
AT THE UNIVERSITY PRESS GLASGOW

PREFACE

THIS text-book is divided into three parts: I. The Right-Angled Triangle; II. The General Triangle and Mensuration; III. The General Angle and Compound Angles. This corresponds to the Matriculation and School Certificate stage. A further volume will deal with Higher Certificate and Scholarship work.

The authors believe that the principles of Trigonometry are most easily grasped if the numerical work is at first of a simple nature; the less time that is required for purely arithmetical computation, the more time is available for illustrating the extensive applications of the subject. The material of Part I. has been so arranged that it may be taken very early in the school course; it assumes nothing more than a knowledge of decimals, simple ratios, and the ideas of drawing to scale. The numerical work is so simple that the use of logarithms is not required. Part II. can best be taken concurrently with Mensuration in Arithmetic and the Properties of Areas in Geometry.

Diagrams have been used to illustrate examples to a much greater extent than is customary; it has thus been possible to introduce an abundance and variety of examples, which make the subject-matter interesting, without burdening the pupil with tedious and complicated verbal descriptions.

The early chapters include supplementary exercises containing harder applications of the elementary principles; these

should, in general, be reserved for a second reading, but may be utilised to keep the quicker pupils in a class profitably occupied while the others are working through the straightforward applications.

The educational value of Trigonometry lies largely in its manifold practical applications and in problems which test insight rather than technique. But progress in later work is impossible without a considerable amount of skill in manipulation, so that a substantial number of examples have been inserted for drill purposes. These are straightforward, devoid of trimming, and are designed solely for the purpose of securing methodical arrangement of the work and facility in handling trigonometrical expressions.

The book-work and illustrative examples throughout the book have been set out in the way in which the pupil would normally be expected to write them out in an examination; but notes have frequently been appended to the work in order to suggest points which may usefully be emphasised in teaching.

Methods of solving the general triangle by division into right-angled triangles have been omitted; the authors consider that it is a wrong policy to teach a method which will shortly be superseded, and with the modern emphasis on the use of formulae in Algebra there should be little risk of pupils applying the sine and cosine formulae without understanding them. Further, every text-book on Arithmetic or Algebra now includes a chapter on Logarithms, which is usually taken comparatively early in the school course. It has therefore seemed unnecessary to add a similar chapter to this volume; a brief chapter only has been included to explain the methods of using the logarithm-tables of the trigonometrical ratios. On the other hand, the treatment of mensuration is fairly complete, partly because many of its practical applications involve the use of Trigonometry and partly because it is valuable in showing how many of the formulae of Mensuration are simplified by the use of radian measure.

Geometrical proofs of the necessary half-angle formulae have been added at the end of Chapter IX., so that, if desired, these formulae can be used, instead of the cosine formulae, for the solution of triangles.

The treatment of the General Angle has been based on the idea of coordinates. This is undoubtedly advantageous now that graphs are included early in the school course, so that the pupil is familiar with the sign conventions employed. This treatment leads to a very simple proof of the addition theorem, valid for angles of any magnitude; the authors are indebted to Professor R. S. Heath for his kind permission to include this proof, which was first given in his text-book on *Elementary Trigonometry*.

As some teachers may prefer to keep to the more usual elementary method of proving this theorem, it has been included as an alternative, but in the opinion of the authors Professor Heath's method is much to be preferred, both for its intrinsic interest and the ease with which it demonstrates the truth of the theorem for angles of any magnitude. The proof by methods of projection was considered too difficult for inclusion at this stage.

Part I. covers the syllabus of the Scottish Leaving Certificate (lower grade). Parts I. and II. together cover the syllabus of the Northern Universities Matriculation and School Certificate, the Oxford Junior Local and the Oxford School Certificate. Parts I.-III. cover the syllabus of the Scottish Leaving Certificate (higher grade), the Cambridge Junior Local, the Cambridge School Certificate, the London Matriculation and the additional mathematics of the Oxford School Certificate, the Cambridge School Certificate and the Oxford and Cambridge Joint Board School Certificate, and that of the Central Welsh Board.

C. V. D.
R. M. W.

October, 1926.

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CHAPTER I.

THE TANGENT OF AN ANGLE.

Historical Note. Numerical Trigonometry was originally devised to meet the needs of the astronomer. The first ideas may be traced as far back as the time of Ahmes, about 1700 B.C., but the earliest systematic treatment is attributed to an astronomer, Hipparchus (150 B.C.), who not only constructed the equivalent of a Table of natural sines but also investigated right-angled spherical triangles. Progress was slow owing to the absence of any suitable notation. Some advance was made by the Arabic School of Bagdad between 800 A.D. and 1400 A.D., the first purely trigonometrical treatise being written by a Persian in the thirteenth century. Knowledge of what the Arabs had done gradually reached Europe through Spain: and by the sixteenth century English mathematicians had obtained a general acquaintance with the methods of plane and spherical *numerical* Trigonometry, while about this time the six ratios received their standard names. The construction of Tables had naturally attracted the attention of mathematicians and astronomers from early times. The most famous of these is the *Opus Palatinum*, compiled by Rhaeticus and a number of assistants and published in 1596; it gives all six ratios at $10''$ intervals to ten decimal places. In the seventeenth century progress became rapid, elementary algebra was assuming its modern form, and this invention of a simple symbolic notation transformed Trigonometry into an analytical subject. Newton's expansions for $\sin nx$ and $\cos nx$ date from 1676, De Moivre's theorem probably from 1707, De Lagny's expansion for $\tan nx$ from 1710, and Lambert's hyperbolic functions from 1760.

The practical application of Trigonometry to the problems of surveying was an afterthought; one of the earliest books

dealing with this aspect is the *Practica Geometriae* of Leonardo of Pisa (1220 A.D.). The reader has already learnt in his elementary geometry how to apply the method of scale-drawing to problems in surveying, the data for which are obtained by using a chain to measure lengths and a theodolite to measure angles both in a vertical and in a horizontal plane. Thales (600 B.C.) had made use of the same idea, the principle of similar figures, to find the height of the pyramids by measuring the lengths of their shadows. By the aid of Trigonometry such problems may now be solved—and to a higher degree of accuracy—by calculation, but the theory is based on the same principles.

Angles. The existing method of measuring angles is modern. In early days astronomers took a circle of some convenient fixed radius and divided the circumference into a number of equal arcs, and worked with these arcs where we now work with angles, that is to say, they measured the length of an arc where we measure the angle standing at the centre on that arc, and they measured the half-chord cutting off an arc where we measure the sine of half the angle at the centre standing on that arc. Whereas we divide four right angles into 360 degrees. the Greeks in the time of Ptolemy (85-165 A.D.), divided the circumference into 360 equal arcs, each arc being called a degree and regarded as the unit measure; they then called $\frac{1}{60}$ of a degree a first part (Latin, *pars minuta prima*, hence our name "minute"), and $\frac{1}{3600}$ of a degree a second part (Latin, *pars minuta secunda*, hence our name "second"). This is called the *sexagesimal measure* of angles:

1 *degree* = 60 *minutes* (60′); 1 *minute* = 60 *seconds* (60″).

The reader is reminded of the following definitions:

(1) Two angles are said to be *complementary* if their sum is 90°.

(2) Two angles are said to be *supplementary* if their sum is 180°.

(3) **Bearings.** There are two principal methods of indicating the direction of any point P from a given point or origin O *in the same horizontal plane.*

(*a*) **The surveyor's method.** The direction of a *horizontal* line OP is given in terms of the cardinal directions N.. E., S., W.; thus, if the direction OP is given as N. 53° E., a man standing at O facing due North and then turning through 53° towards the East is now facing P. Similarly, if the direction OQ is S. 19° E., or, in other words, if the "*bearing*" of Q from O is

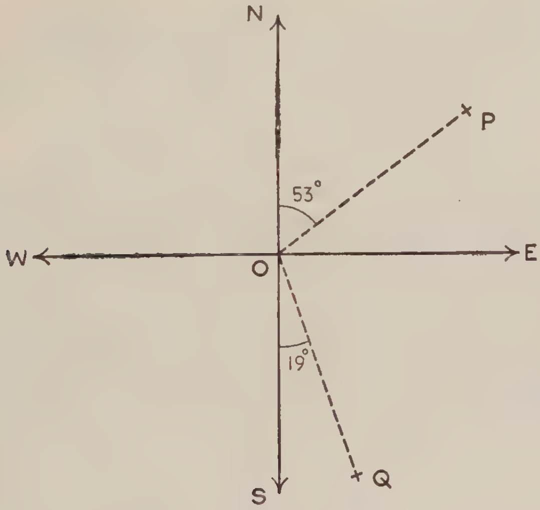


FIG. 1.

S. 19° E., a man standing at O facing due South and then turning through 19° towards the East is now facing Q.

Note. In using this method bearings should always be reckoned from the North or from the South, not from the East or West. Thus OP should be described as N. 53° E., *not* E. 37° N.; similarly one should say S. 78° W., *not* W. 12° S. This practice is adopted to avoid the possibility of mis-reading.

(*b*) **The soldier's method.** In the army all bearings are given from the geographical or the true North. The "true *bearing*" of a line is the angle the line makes with the true

North, the angle being measured in a clock-wise direction, *i.e.* from the North through East and South.

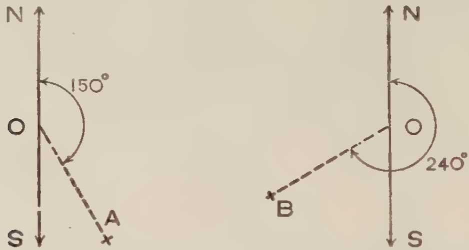


FIG. 2 (i). FIG. 2 (ii).

Thus the bearing of A from O in Fig. 2 (i) would be given as a bearing of 150° by this method, and not as S. 30° E.; and the bearing of B from O in Fig. 2 (ii) would be given as a bearing of 240°, and not as S. 60° W.

Note. Complications are introduced in practice from the fact that "True" North, "Magnetic" North, and "Grid" North differ; for details reference should be made to books on practical surveying and military manuals on map-reading.

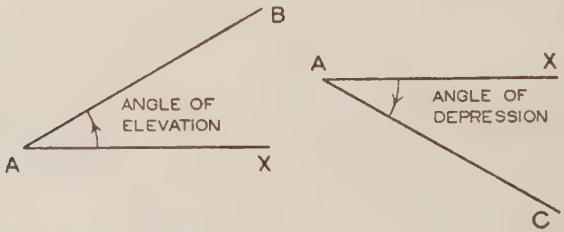


FIG. 3.

(4) **Angles of Elevation and Depression.**

If B is an object above A, *the angle of elevation* of B from A is the angle which AB makes with the horizontal plane AX through A.

If C is an object below A, *the angle of depression* of C from A is the angle which AC makes with the horizontal plane AX through A.

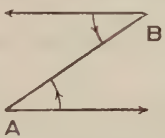


FIG. 4.

Note. The angle of elevation of B from A is equal to the angle of depression of A from B, for these are alternate angles.

EXERCISE I. a.

(Oral.)

1. How many degrees are there (i) in 3 right angles, (ii) in half a right angle, (iii) in $\frac{2}{3}$ right angle ?

2. What is (i) the complement of 20°, (ii) the supplement of 25°, (iii) the supplement of 54° 20′, (iv) the complement of 72° 50′, (v) the supplement of 137° 25′ ?

3. What is the third angle of a triangle if two of the angles are (i) 90°, 40° ; (ii) 90°, 26° 30′ ; (iii) 90°, 63° 50′ ; (iv) 27°, 66° ; (v) 108° 15′, 39° 55′ ?

4. How many degrees are there between (i) N.E. and S.E. ; (ii) S. 10° W. and S. 48° E. ; (iii) S. 10° W. and N. 50° W. ; (iv) a bearing of 100° and a bearing of 210° ; (v) a bearing of 20° and a bearing of 350° ?

5. Give the following bearings in the army form : (i) N. 70° E. ; (ii) S. 10° E. ; (iii) S. 20° W. ; (iv) N. 50° W.

6. Give the following true bearings in the surveyor's form : (i) 50° ; (ii) 210° ; (iii) 358° ; (iv) 110°.

7. What is the bearing of O from A if the bearing of A from O is (i) N. 10° E. ; (ii) S. 14° E. ; (iii) 15° ; (iv) 310° ?

8. The angle of elevation of Q from P is observed to be 18° 45′. What is the observation of P from Q ?

9. The angle of depression of a boat from the top of a cliff is observed to be 15° 27′. What is the elevation of the top of the cliff as seen by a man in the boat ?

(Written.)

10. Express in degrees, minutes, seconds, correct to the nearest second, (i) 28·372 degrees ; (ii) $\frac{2}{7}$ right angle ; (iii) 10,000 seconds ; (iv) 252·4 minutes.

11. Express as a decimal of a degree correct to 3 places of decimals, (i) $25° 35' 25''$; (ii) $108° 17' 20''$.

12. Through what angle does the earth turn in one minute of time?

13. Through what angle does the hour hand of a clock turn in one minute of time?

14. Cape Town has latitude $33° 40'$ S. and longitude $18° 30'$ E. Cologne has latitude $50° 55'$ N. and longitude $7°$ E. What is their difference of latitude and longitude?

15. From A the bearing of B is N. $80°$ E. and the bearing of C is N. $50°$ E.; also B and C are equidistant from A. What is the bearing of C from B?

Similar triangles. The subject of Trigonometry depends in the first instance upon the fact that two equiangular triangles have their corresponding sides proportional. If a number of triangles are drawn with the same set of angles they will all have the same shape. The following experiment can be performed by a class of pupils:

"Draw a triangle ABC having $\angle A = 40°$, $\angle B = 90°$, $\angle C = 50°$. Measure AB, BC. Work out the value of the ratio $\frac{BC}{AB}$."

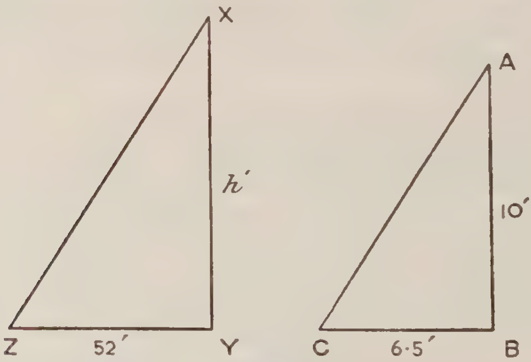


FIG. 5.

The triangles thus drawn will differ in *size*, but they should all have the same *shape* and, subject to errors of experiment,

the values obtained for the ratio $\frac{BC}{AB}$ should all be the same, viz., about 0·84.

Example I. A pole 10′ high casts a shadow $6\frac{1}{2}'$ long; at the same time the shadow of a church tower is 52′ long. What is the height of the tower?

Δs ABC, XYZ represent the triangles formed by the pole and tower and their shadows (see Fig. 5). Since the sun's rays strike the earth at the same angle, $\angle C = \angle Z$. Also $\angle B = \angle Y = 90°$.

$$\therefore \text{ the } \Delta\text{s are equiangular and similar.}$$

$\therefore$ If XY, the height of the tower, is h ft.,

$$\frac{h}{52} = \frac{10}{6\frac{1}{2}};$$

$$\therefore \; h = \frac{10 \times 52}{6\frac{1}{2}} = 80;$$

$\therefore$ the height of the tower is **80 ft.**

EXERCISE I. b.

1. Draw two triangles ABC, PQR, having

$\angle A = \angle P = 25°$, $\angle B = \angle Q = 90°$, $AB = 4$ in., $PQ = 3$ in.

Measure BC and QR and find the values of the ratios $\frac{BC}{AB}, \frac{QR}{PQ}$.

2. Draw two triangles ABC, XYZ having

$\angle A = \angle X = 32°$, $\angle B = \angle Y = 80°$, $AB = 10$ cm., $XY = 3$ in.

Measure BC in cm. and YZ in inches, and find the values of the ratios $\frac{BC}{AB}, \frac{YZ}{XY}$.

3. If, in Fig. 6, BC is drawn 2 cm. long, it is found that AC is 1·20 cm. long.

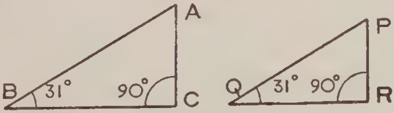


FIG. 6.

What is PR if (i) $QR = 5$ cm.; (ii) $QR = 3$ in.?
What is QR if (i) $PR = 3·6$ cm.; (ii) $PR = 6$ in.?
What are the values of $\frac{AC}{CB}$ and $\frac{PR}{RQ}$?

4. ABC, PQR are two triangles, right-angled at C and R, and such that the angles at B, Q are each 58°. If BC is 5 cm., it is found by measurement that CA is 8 cm.

What is PR if (i) QR = 7 cm.; (ii) QR = 5 in.?
What is QR if (i) PR = 6 cm.; (ii) PR = 1 ft.?
What are the values of $\frac{AC}{CB}$ and $\frac{PR}{RQ}$?

5. When the shadow of a vertical stick 3 ft. high is 3 ft. 9 in. long the shadow of a tower is 90 ft. long. What is the height of the tower?

6. A halfpenny (diameter 1 inch) placed at a distance of 3 yds. from the eye will just obscure the disc of the sun or moon. Taking the distance of the sun as 93 million miles, find its diameter. Taking the diameter of the moon as 2160 miles, find its distance.

7. How far in front of a pinhole camera must a man 6 ft. high stand in order that a full-length photograph may be taken on a film 3 in. high and $2\frac{1}{2}$ in. from the pin-hole?

8. Two scale-drawings are made of a rectangular court 100 yd. long, 60 yd. wide, one on a scale of 10 yd. to the cm., the other on a scale of 20 yd. to the inch. What are the dimensions of the drawings? Are they the same shape?

9. A path 1 yd. wide runs all round a rectangular lawn 20 yd. long, 15 yd. wide. Is the rectangle formed by the outer edge of the path the same shape as the lawn?

10. The radius of the base of a cone is 8″ and its height is 15″. What is the radius of a section parallel to the base and 6″ from it?

The tangent of an angle. Example I. on p. 7 and the examples in Exercise I. b. are illustrations of the fact that if

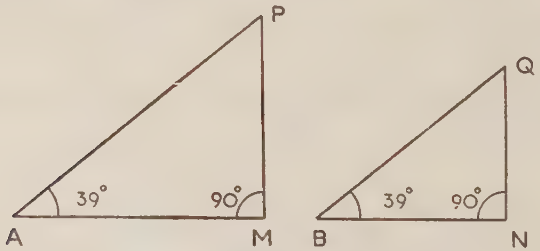


FIG. 7.

two triangles are of the same shape the ratio of a pair of sides in one triangle is equal to the ratio of the pair of corresponding sides in the other.

For instance, the $\triangle$s APM, BQN in Fig. 7 are the same shape, and the ratio $\frac{MP}{AM}$ is equal to the ratio $\frac{NQ}{BN}$.

The reader can draw two triangles of this shape, making, for instance, in one $AM = 10$ cm., and in the other $BN = 2''$.

He should find $MP = 8{\cdot}1$ cm. and $NQ = 1{\cdot}62''$, and obtain the results

$$\frac{MP}{AM} = \frac{8{\cdot}1}{10} = 0{\cdot}81, \quad \text{and} \quad \frac{NQ}{BN} = \frac{1{\cdot}62}{2} = 0{\cdot}81.$$

The value of this ratio depends only on the fact that in both these triangles one angle is 39° and one angle is 90°.

The ratio $\frac{MP}{AM}$ in Fig. 7 is called the **tangent of the angle** PAM, and is written **tan** $\angle$PAM, or tan 39°, since $\angle PAM = 39°$.

The general statement may now be made :

If a perpendicular is drawn from **any** *point in* **either** *arm of an angle to the other arm,* **the tangent of the angle** ($\angle$PAM)

$$= \frac{\textbf{side opposite angle}}{\textbf{side adjacent to angle}}, \ i.e. \ \frac{MP}{AM}.$$

The approximate value of the tangent of an angle may be found by measurement; for instance, if Fig. 7 is drawn accurately with $AM = 10$ cm., it will be found that

$$\tan 39° = \frac{MP}{AM} = \frac{8{\cdot}1}{10} = 0{\cdot}81.$$

But the tangents of angles have been calculated once for all by mathematicians, and have been entered in books of Tables from which they can be obtained when required. A book of seven-figure Tables will give $\tan 39° = 0{\cdot}8097840$. Seven-figure Tables are required by astronomers, but for most practical purposes four-figure Tables are sufficiently accurate ; they will give

$$\tan 39° = 0{\cdot}8098.$$

Use of tangent Tables. A book of four-figure Tables (see p. 10) gives the tangents of angles from 0° to 90° at intervals of 6

minutes, and by means of the *difference-columns* at the side it is possible to find the values for intervals of 1 minute.

Extract from Table of Natural Tangents.

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
50	1·1918	1960	2002	2045	2088	2131	2174	2218	2261	2305	7	14	22	29	36

This extract shows that $\tan 50° = 1{\cdot}1918$ (to 4 places).

Similarly, $\tan 50°\ 12' = 1{\cdot}2002$, etc.

To find $\tan 50°\ 44'$, we say

$$\tan 50°\ 42' = 1{\cdot}2218$$
$$\text{Difference for } 2' = 0{\cdot}0014\ ;$$
$$\therefore\ \tan 50°\ 44' = 1{\cdot}2232.$$

Note. The difference columns can only give average differences, and the fourth decimal place is not therefore reliable. The results, as in any work with four-figure logarithms, are only approximate. *Four* figures should always be retained throughout the working of an example, but the *final result* should be given correct to *three* significant figures as a general rule.

Note. The "tangent" of an angle was first used by Abul-Wefa (940-998 A.D.); he also formulated the relations

$$\tan\theta = \frac{\sin\theta}{\cos\theta}\ ; \quad \sec^2\theta = 1 + \tan^2\theta\ ; \quad \operatorname{cosec}^2\theta = 1 + \cot^2\theta\ ;$$
$$\sin 2\theta = 2\sin\theta\cos\theta.$$

The relation $\sin^2\theta + \cos^2\theta = 1$ was recognised by Ptolemy.

Example II. Find, by drawing, approximate values of $\tan 12°$, $\tan 24°$, $\tan 36°$, $\tan 48°$.

Draw a circle of unit radius, centre O, diameter AOB.

(*Note.* The reader should draw his own figure and take 1 decimetre as his unit. Figure 8 represents part of a circle of radius 1 inch.)

Draw the tangent at A to the circle and draw lines OP, OQ, OR, OS cutting the tangent at P, Q, R, S, and such that

$$\angle AOP = 12°, \quad \angle AOQ = 24°, \quad \angle AOR = 36°, \quad \angle AOS = 48°.$$

The angle at A is 90°;

$$\therefore \tan 12° = \frac{AP}{OA}.$$

By measurement AP = 0·21 in.; also OA = 1 in.;

$$\therefore \tan 12° = \frac{0{\cdot}21}{1} \text{ approx.} = 0{\cdot}21 \text{ approx.}$$

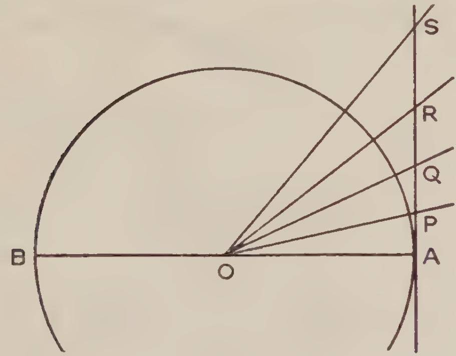


FIG. 8.

Similarly by measurement,

$$AQ = 0{\cdot}45 \text{ in.}, \quad AR = 0{\cdot}73 \text{ in.}, \quad AS = 1{\cdot}11 \text{ in.}$$

$$\therefore \tan 24° = \frac{0{\cdot}45}{1} = 0{\cdot}45 \text{ approx.},$$

$$\tan 36° = \frac{0{\cdot}73}{1} = 0{\cdot}73 \text{ approx.},$$

$$\tan 48° = \frac{1{\cdot}11}{1} = 1{\cdot}11 \text{ approx.}$$

If then the radius is unity, the length (or rather the number of units in the length) cut off on the tangent, as drawn in Fig. 8, represents the tangent of the corresponding angle. This is the reason for the name chosen for this particular ratio. But it is important to notice that the tangent of an angle is a *ratio*, *i.e.* a pure number independent of any unit of length used in finding or applying it. Further, the tangent of an angle is not

directly proportional to the size of the angle. Thus tan 24° is more than twice tan 12°, and tan 48° is more than twice tan 24°: and the nearer the angle approaches 90° the larger the value of its tangent becomes; by taking an angle sufficiently near 90° we can make the value of its tangent as large as we please.

Example III. Given a triangle ABC such that $\angle ACB = 90°$, $\angle ABC = 56°$, $AC = 6''$, calculate BC.

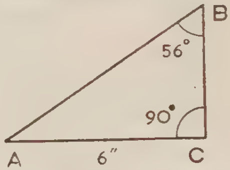


FIG. 9.

Since $\angle ABC = 56°$, $\angle BAC = 90° - 56° = 34°$.

Then $$BC = AC \times \frac{BC}{AC} = 6 \tan 34°$$
$$= 6 \times 0{\cdot}6745 = 4{\cdot}047 = \mathbf{4{\cdot}05\ inches}\ \text{(to 3 figures)}.$$

Note. We might also argue as follows:

$$\frac{AC}{BC} = \tan 56°; \quad \therefore \frac{BC}{AC} = \frac{1}{\tan 56°};$$
$$\therefore BC = \frac{6}{\tan 56°} = \frac{6}{1{\cdot}4826}$$
$$= 4{\cdot}047 = \mathbf{4{\cdot}05\ inches}\ \text{(to 3 figures)}.$$

The first method is obviously simpler. The reader should note that $\tan 56° = \frac{1}{\tan 34°}$, and that in general the tangent of any angle is equal to the reciprocal of the tangent of the complementary angle.

Notation. If ABC is a triangle, it is usual to denote the lengths of the sides BC, CA, AB by a, b, c respectively, and the magnitudes of the opposite angles by A, B, C respectively.

Thus in the above example $b = 6''$, $B = 56°$, $C = 90°$.

Example IV. Find the angle of elevation of the top of a tower 60 ft. high as seen from a point on the level ground distant 240 yds. from the foot of the tower.

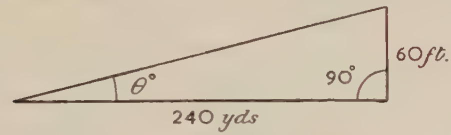


FIG. 10.

If the angle of elevation required is $\theta°$, then

$$\tan\theta° = \frac{60 \text{ ft.}}{240 \text{ yds.}}$$

$$= \frac{60}{240 \times 3} = \frac{1}{12} = 0{\cdot}0833;$$

$$\therefore \text{ from the Tables, } \theta° = \mathbf{4°\ 46'}.$$

Note. The tables show that $\tan 4°\ 42' = 0{\cdot}0822$,
and that $\tan 4°\ 48' = 0{\cdot}0840$.

In the above example, $\tan\theta° = 0{\cdot}0833$. Since

$$0{\cdot}0833 - 0{\cdot}0822 = 0{\cdot}0011,$$

we look at the difference columns and read off the number of minutes which correspond to a difference of 11. Here the nearest number is 4′. We therefore say $\theta° = 4°\ 42' + 4' = 4°\ 46'$.

EXERCISE I. c.

[All results involving calculation should be given correct to three significant figures, unless otherwise stated (see note on p. 10).]

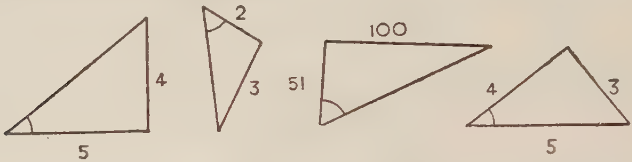
1. Find by drawing the values of tan 20°, tan 40°, tan 45°, tan 50°, tan 60°, tan 75°. (It saves time to use squared paper.) Then find their values from the Tables.

2. Find by drawing (preferably using squared paper) the angles whose tangents are $\frac{1}{4}$, 0·9, 1·6, 2·3. Then find these angles from using the Tables.

3. Use the Tables to write down the values of tan 62°, tan 32° 24′, tan 15° 45′, tan 63° 43′.

4. Use the Tables to write down the angles whose tangents are 0·4245, 2·9042, 0·2754, 3·0061, 28·64, 0·2536, 1·2016, 0·8922.

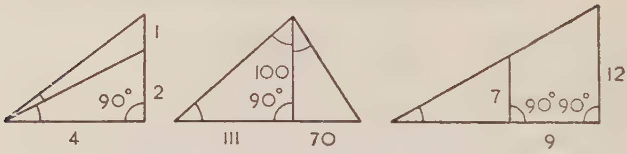
5. Find the marked angles in Fig. 11, (i), (ii), (iii), (iv), given that the triangles are right-angled.



(i) (ii) (iii) (iv)

FIG. 11.

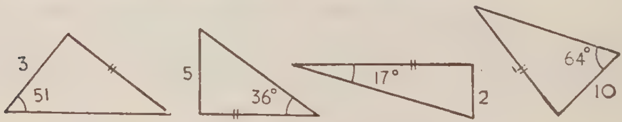
6. Find the marked angles in Fig. 12, (i), (ii), (iii).



(i) (ii) (iii)

FIG. 12.

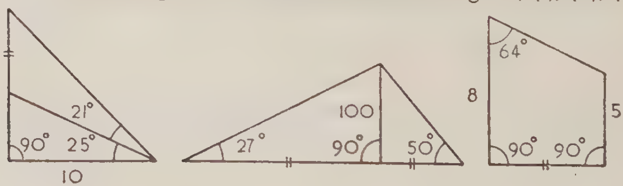
7. Find the lengths of the marked sides in Fig. 13, (i), (ii), (iii), (iv), given that the triangles are right-angled.



(i) (ii) (iii) (iv)

FIG. 13.

8. Find the lengths of the marked sides in Fig. 14, (i), (ii), (iii).



(i) (ii) (iii)

FIG. 14.

9. ABC is an equilateral triangle of side 2 inches; AD is perpendicular to BC (Fig. 15). Use Pythagoras to prove that $AD = \sqrt{3}$ inches. Then *calculate* tan 60° and tan 30°, and compare with the Tables.

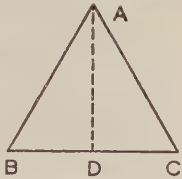


FIG. 15.

10. Find from a suitable figure the value of tan 45°.

11. ABC is an isosceles triangle with AB = AC; AD is drawn perpendicular to BC.

(i) If $B = 37°$, $a = 6$ cm., calculate AD.

(ii) If $B = 42°$, $AD = 5$ cm., calculate BC.

(iii) If $A = 96°$, $a = 4$ in., calculate AD.

12. From a point on the ground, 100 yards away from a tower, the angle of elevation of the top of the tower is 33° 30′. Find the height of the tower.

13. From the top of a cliff 250 feet high the angle of depression of a boat is 17°. Find the distance of the boat from the cliff.

14. The shadow of a vertical pole 12 ft. high is 17 ft. 4 in. long. What is the altitude of the sun?

15. The vertical angle of a cone is 102°, and the diameter of its base is 5 inches. What is its height?

16. A ladder leaning against a vertical wall makes an angle of 21° with the wall; the foot of the ladder is 5 ft. from the wall. How high up the wall does the ladder reach?

17. A man starts from O and walks 2 miles East and then $\frac{1}{2}$ mile South. What is his bearing from O? What is his new bearing from O when he walks another half mile South?

18. A chest of drawers, 3 ft. high, stands in an attic with a roof sloping down to the floor. If the chest can only just stand 2 ft. from the edge of the room, find the slope of the roof.

19. What is the angle of elevation of the top of a spire 240 ft. high from a point on the ground 200 yd. from the foot of it?

20. The pole of a bell tent is 8 ft. high, and the diameter of the base of the tent is 14 ft. What angle does the slant side of the tent make with the ground?

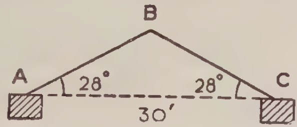


FIG. 16.

21. Using tables, find A, B and A + B, if (i) $\tan A = 2$, $\tan B = \frac{1}{2}$, (ii) $\tan A = \frac{4}{5}$, $\tan B = \frac{5}{4}$.

22. Fig. 16 represents the roof of a villa; find the height of the ridge B of the roof above the top of the walls.

23. The diagonals of a rhombus are 6 in., 4 in. long. What are the angles of the rhombus ?

24. One angle of a rhombus is 144° ; the shorter diagonal is 5 cm. long. Find the other diagonal.

25. The vertical angle of an isosceles triangle is 45° and the base is 6 in. long. Find the area of the triangle.

26. A cricket ball is rolled in a straight line down the pitch from immediately alongside one of the stumps at one end of the pitch. Find within what angle its direction of motion lies if it does not miss the wickets at the other end. Take the diameter of the ball as 3 inches and the extreme width of the stumps as 8 inches.

27. A chord of a circle is 6 cm. long and subtends an angle of 103° 30′ at the centre. Find its distance from the centre.

28. The steps of a staircase are 10 inches deep and 6 inches high, What angle does the bannister rail make with the horizontal ?

29. The greatest and least heights of a lean-to shed are 10 ft. and 7 ft. 3 in. ; the floor is 12 ft. wide. Find the slope of the roof.

30. Fig. 17 represents the section of a railway cutting ; the base BC is horizontal and 15 ft. wide ; the tops A, D are each 18 ft. above BC. Find AD.

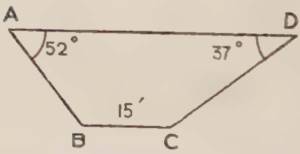


FIG. 17.

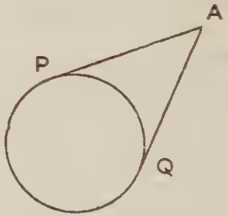


FIG. 18.

31. AP, AQ are tangents to a circle of radius 4 inches ; $\angle PAQ = 41°$. Find AP.

32. A map shows a straight road crossing two contour levels 100 ft., 200 ft. at P, Q. The length of PQ is 1·2 inches, and the scale of the map is 4 inches to the mile. What average angle does the road make with the horizontal ?

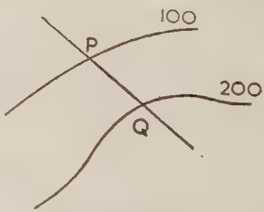


FIG. 19.

33. A circle is inscribed in an equilateral triangle of side 6 inches. What is its radius ?

34. AB = 3 cm., BC = 7 cm., AP bisects ∠ BAC. Find PC.

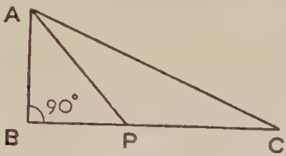


FIG. 20.

The following exercise may be reserved for a second reading.

Example V. ABC is a triangle such that

$$BC = 4 \text{ in.}, \quad \angle ABC = 61°, \quad \angle ACB = 68°.$$

A circle is drawn to touch AB produced, AC produced and BC. Calculate its radius.

Let K be the centre of the circle, so that KB, KC bisect ∠s DBC, ECB; let BC touch the circle at N; let KN = r inches.

$$\angle DBC = 180° - 61° = 119°;$$

$$\angle KBN = 59° \, 30';$$

$$\therefore \angle BKN = 90° - 59° \, 30' = 30° \, 30'.$$

Similarly

$$\angle KCN = \tfrac{1}{2}(180° - 68°) = 56°,$$

$$\therefore \angle NKC = 90° - 56° = 34°.$$

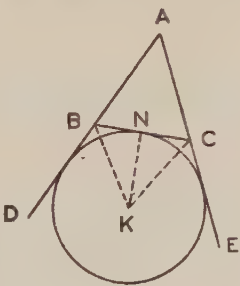


FIG. 21.

$$\therefore BN = \frac{BN}{NK} \times NK = r \tan 30° \, 30'$$

and $$NC = r \tan 34°;$$

$$\therefore r \tan 30° \, 30' + r \tan 34° = BN + NC = BC = 4;$$

$$\therefore r(0{\cdot}5890 + 0{\cdot}6745) = 4;$$

$$\therefore 1{\cdot}2635r = 4; \quad \therefore r = \frac{4}{1{\cdot}263} = 3{\cdot}17 \text{ inches.}$$

EXERCISE I. d.

1. Two men, one due North and the other due South of a tower, measure the angles of elevation of the top of its spire as 28° and 37°; the height of the spire is 120 feet. How far apart are the men ?

2. The shadow of a vertical pole is 10 ft. long when the sun's elevation is 35°. What is the length of the shadow when the sun's elevation is 25° ?

3. The angle of elevation of the top of a tower from a point on the ground 120 yards from its foot is 21° 48′. What will it be from a point on the ground 20 yards nearer the tower ?

4. The sides of a rectangle are 4, 5 inches long. What is the angle between the diagonals ?

5. Find a value of θ if $\tan \theta° = 3 \tan 20°$.

6. Simplify (i) $\tan 20° \times \tan 70°$; (ii) $\tan 34° - \dfrac{1}{\tan 56°}$.

7. From the top of a cliff 300 feet high the angles of depression of two boats in a vertical plane with the observer are 25° 24′, 37° 52′. Find the distance between the boats.

8. A man stands at a distance of 90 ft. from the foot of a tower and observes that the angles of elevation of the top and bottom of a flagstaff on it are 56° and 53° respectively. What is the length of the flagstaff ?

9. A conical funnel, vertical angle 52°, rests inside a glass of height 7 inches and diameter 3 inches, internal measurements. Find the height of the apex of the funnel above the base of the glass.

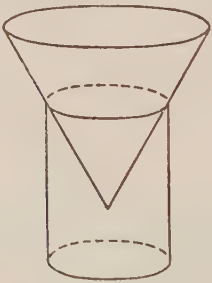


FIG. 22.

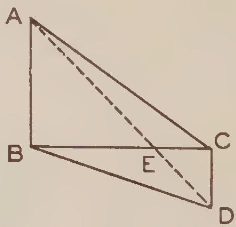


FIG. 23.

10. In Fig. 23,

$\angle ABC = 90° = \angle BCD$, $\angle ACB = 41° 27'$, $\angle CBD = 32° 44'$, $BC = 10$ cm. Calculate AB, CD and $\angle AEB$.

11. The base of a tank is 2 ft. square, and contains water to a depth of 1 ft. It is tilted about one edge as shown, through 15°. What is the length of AP ?

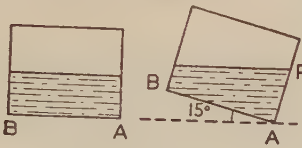


FIG. 24.

12. From halfway up a tower the angle of depression of a mark on the ground is 52° 27′. What will it be from the top of the tower ?

13. Two circles of radii 5, 3 cm. are drawn touching a line at points A, B, 7 cm. apart; the other external common tangent PQ cuts AB at T. Calculate $\angle$ATP.

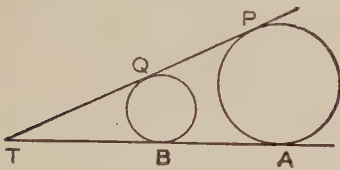


FIG. 25.

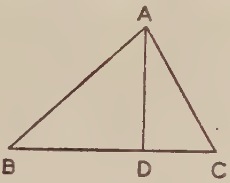


FIG. 26.

14. $\angle$BAC = 90° = $\angle$ADB ; BD = 12 in., DC = 6 in. Find $\angle$ABC.

15. O is the centre and C is the apex of a thin hemispherical shell or bowl. When suspended from a point A of the rim the shell hangs so that the mid-point of OC is vertically below A. What angle will AC make with the vertical ?

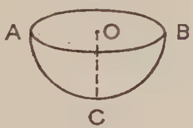


FIG. 27.

16. ABC is a triangle such that $\angle$ABC = 37° 15′, $\angle$ACB = 59° 40′, BC = 8 cm. ; the perpendicular bisector of BC cuts BA, CA produced at P, Q. Find the length of PQ.

17. ABC is a triangle such that

$$BC = 8 \text{ cm.}, \quad \angle ABC = 46^\circ, \quad \angle ACB = 62^\circ.$$

A circle is inscribed in the triangle, *i.e.* touches the three sides. Calculate its radius.

18. A, B are points on opposite sides of a street 32 ft. wide, each at a height of 25 ft. above the street. Lights are attached to points P, Q, R on a wire suspended from A and B as shown, and are arranged so as to be at equal *horizontal* intervals across the street. Find the heights of P, Q, R above the ground.

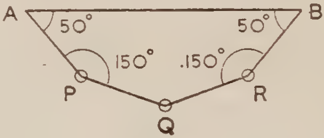


FIG. 28.

19. ABC is a triangle such that $\angle ABC = 24^\circ$, $\angle ACB = 110^\circ$; O is the mid-point of BC and AD is drawn perpendicular to BC produced. Show that $DO = \frac{1}{2}(DB + DC)$. Hence prove that

$$\tan OAD = \tfrac{1}{2}(\tan BAD + \tan CAD),$$

and calculate $\angle AOC$.

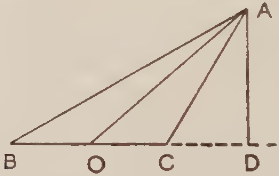


FIG. 29.

20. ABC is a triangle such that

$$\angle ABC = 56^\circ, \quad \angle ACB = 42^\circ, \quad BC = 5 \text{ inches}.$$

Calculate the length of the perpendicular from A to BC.

CHAPTER II.

THE SINE AND COSINE.

The sine of an angle. We saw in Fig. 7, p. 8, that if the angles at A, B are equal, and if perpendiculars are drawn from

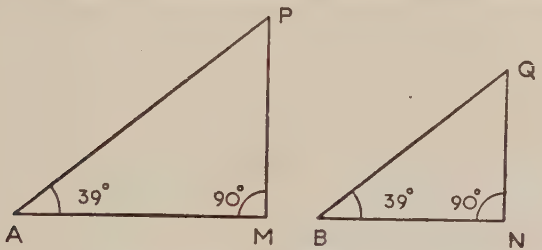


FIG. 7.

points P, Q on *either* arm of the angle to the other arm, the two right-angled triangles so obtained are the same shape.

Consequently $$\frac{MP}{AP}=\frac{NQ}{BQ}.$$

Therefore the value of the ratio $\frac{MP}{AP}$ does not depend on the length of AP, but only on the size of the angle MAP.

We can test this approximately by measurement :

$$MP=2{\cdot}65 \text{ cm.},\quad AP=4{\cdot}2 \text{ cm.},\quad \frac{MP}{AP}=\frac{2{\cdot}65}{4{\cdot}2}=0{\cdot}63, \text{ approx.}$$

$$NQ=0{\cdot}79 \text{ in.},\quad BQ=1{\cdot}25 \text{ in.},\quad \frac{NQ}{BQ}=\frac{0{\cdot}79}{1{\cdot}25}=0{\cdot}63, \text{ approx.}$$

The ratio $\frac{MP}{AP}$ in Fig. 7 is called the **sine of the angle** MAP,

and is written **sin** MAP or sin 39°, since $\angle$MAP = 39°. We may state this as follows:

If a perpendicular is let fall from any point on **either** *arm of an angle to the other arm,*

the sine of the angle ($\angle$MAP) $= \dfrac{\textbf{side opposite angle}}{\textbf{hypotenuse}}$, **i.e.** $\dfrac{MP}{AP}$;

or, more shortly, for any angle $\theta°$, $\sin\theta° = \dfrac{\text{opposite}}{\text{hypotenuse}} = \dfrac{O}{H}$.

The cosine of an angle. We also know that $\dfrac{AM}{AP} = \dfrac{BN}{BQ}$; therefore the value of the ratio $\dfrac{AM}{AP}$ does not depend on the length of AP, but only on the size of the angle MAP.

The ratio $\dfrac{AM}{AP}$ in Fig. 7 is called the **cosine of the angle** MAP, and is written **cos** MAP or cos 39°, since $\angle$MAP = 39°. We may state this as follows:

If a perpendicular is let fall from any point on **either** *arm of an angle to the other arm,*

the cosine of the angle ($\angle$MAP) $= \dfrac{\textbf{side adjacent to angle}}{\textbf{hypotenuse}}$, **i.e.** $\dfrac{AM}{AP}$;

or, more shortly, for any angle $\theta°$, $\cos\theta° = \dfrac{\text{adjacent}}{\text{hypotenuse}} = \dfrac{A}{H}$.

Summary of definitions. With the notation of Fig. 30 and Fig. 31, we have

$\sin\theta° = \dfrac{\mathbf{O}}{\mathbf{H}}$; $\dfrac{\textbf{opposite}}{\textbf{hypotenuse}}$.

$\cos\theta° = \dfrac{\mathbf{A}}{\mathbf{H}}$; $\dfrac{\textbf{adjacent}}{\textbf{hypotenuse}}$.

$\tan\theta° = \dfrac{\mathbf{O}}{\mathbf{A}}$; $\dfrac{\textbf{opposite}}{\textbf{adjacent}}$.

From the definitions we see that

$$\frac{\sin\theta}{\cos\theta} = \frac{O}{H} \div \frac{A}{H} = \frac{O}{A} = \tan\theta.$$

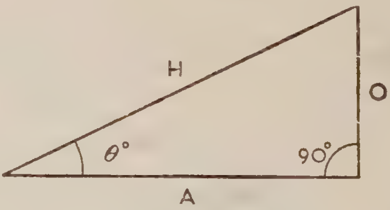


FIG. 30.

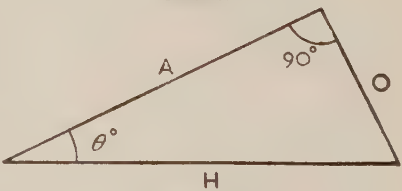


FIG. 31.

Note. (i) The letters O.H M.S. may be used to remember the fact that "**O**pposite over **H**ypotenuse **M**eans **S**ine."

(ii) The reader must accustom himself to the different ways in which a figure can be turned round, cf. Fig. 30 and Fig. 31 above.

Example I. Find, by drawing and measurement, approximate values of

$$\sin 20°, \quad \cos 20°; \quad \sin 40°, \quad \cos 40°; \quad \sin 70°, \cos 70°.$$

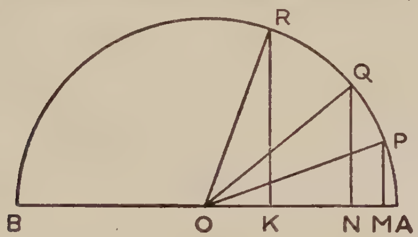


FIG. 32.

Draw a circle of unit radius, centre O, diameter AOB.

(*Note.* The reader should draw his own figure, preferably on squared paper, and take 1 dm. or 5 inches as his unit. Fig. 32 represents part of a circle of radius 1 inch. It is unnecessary to draw more than a quarter of the circle.)

Draw lines OP, OQ, OR cutting the circle at P, Q, R and such that $\angle AOP = 20°$, $\angle AOQ = 40°$, $\angle AOR = 70°$; draw the perpendiculars PM, QN, RK to AB.

By definition,

$$\sin 20° = \sin MOP = \frac{MP}{OP} = \frac{0{\cdot}34 \text{ in.}}{1 \text{ in.}} = 0{\cdot}34, \text{ approx.}$$

Similarly, $$\cos 20° = \frac{OM}{OP} = \frac{0{\cdot}94 \text{ in.}}{1 \text{ in.}} = 0{\cdot}94, \text{ approx.}$$

And from the other necessary measurements we have

$$\sin 40° = \frac{NQ}{OQ} \simeq 0{\cdot}64; \quad \cos 40° = \frac{ON}{OQ} \simeq 0{\cdot}77.$$

$$\sin 70° = \frac{KR}{OR} \simeq 0{\cdot}94; \quad \cos 70° = \frac{OK}{OR} \simeq 0{\cdot}34.$$

Note. (i) The construction used in this Example shows that as the angle θ° increases from 0° to 90°, $\sin\theta^\circ$ *increases steadily* from 0 to 1, since it is represented by $\frac{\mathsf{MP}}{\text{radius}}$, and $\cos\theta^\circ$ *decreases steadily* from 1 to 0, since it is represented by $\frac{\mathsf{OM}}{\text{radius}}$.

(ii) $\sin 40^\circ$ is *less than* twice $\sin 20^\circ$; the value of the sine of an angle is *not* proportional to the angle.

(iii) The values obtained above show that $\sin 20^\circ = \cos 70^\circ$, and $\cos 20^\circ = \sin 70^\circ$. This follows from the fact that the triangles OMP, RKO are congruent; but it is easier to deduce it from Fig. 33.

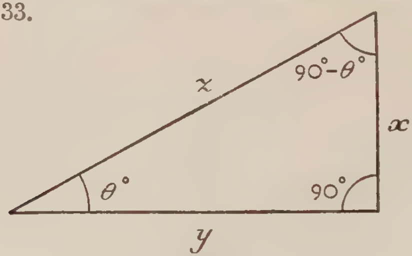


FIG. 33.

By definition, with the notation of Fig. 33,

$$\sin\theta^\circ = \frac{x}{z} = \cos(90^\circ - \theta^\circ),$$

and
$$\cos\theta^\circ = \frac{y}{z} = \sin(90^\circ - \theta^\circ).$$

Hence *the sine of any angle equals the cosine of its complement.*

Use of Tables. The sine-table is used in exactly the same way as the tangent-table (see p. 10). But, in using the cosine-tables, the figures in the difference-column must be *subtracted*, for *the cosine decreases as the angle increases.*

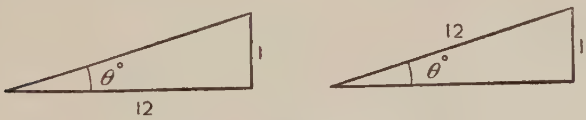
Thus, to find $\cos 53^\circ\ 20'$,

$$\cos 53^\circ\ 18' = 0{\cdot}5976$$
$$\text{Difference for } 2' = 0{\cdot}0005;$$
$$\therefore \cos 53^\circ\ 20' = 0{\cdot}5971.$$

Note. Since the sine of any angle equals the cosine of its complement, and *vice-versa*, the Table of Natural Cosines is obtained by writing the Table of Natural Sines backwards: some books of Tables do not therefore print the values of natural cosines separately.

The name *sine*, or rather its Latin equivalent *sinus*, was first used in the twelfth century, but was not adopted universally till the seventeenth century, when the abbreviation *sin* was first employed (1634); the cosine came into use first in India, about 500 A.D., simply as the sine of the complementary angle, and it was a long time before it received any recognised name of its own; the term *cosinus* was used by Gunter in 1620, and the abbreviation to *cos* was made fifty years later.

Gradient. The statement that the gradient of a road is 1 in 12 is ambiguous.



(i) (ii)

FIG. 34.

It may either mean that the road rises 1 ft. vertically for each 12 ft. measured *horizontally* (Fig. 34 (i)), or that it rises 1 ft. vertically for each 12 ft. measured *along the slope* (Fig. 34 (ii)).

If $\theta°$ is the angle which the road makes with the horizontal,

$$\text{in (i)}, \quad \tan\theta° = \tfrac{1}{12} = 0{\cdot}0833; \quad \therefore\ \theta = 4°\,46';$$
$$\text{in (ii)}, \quad \sin\theta° = \tfrac{1}{12} = 0{\cdot}0833; \quad \therefore\ \theta = 4°\,47'.$$

This example shows that for small slopes the exact meaning is almost immaterial; but for large slopes the precise meaning must be specified. The former meaning ($\tan\theta°$) is commonly attributed in work with graphs; the latter ($\sin\theta°$) is adopted by surveyors and engineers.

Example II. A ladder 10 ft. long leans against a wall and is inclined at 53° to the ground. How far from the wall is the foot of the ladder? How high up the wall does the ladder reach?

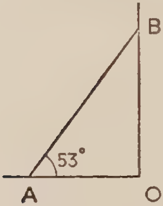


FIG. 35.

AB is the ladder; AB = 10 ft.

$$\text{AO} = \text{AB} \times \frac{\text{AO}}{\text{AB}} = 10 \cos 53° \text{ ft.}$$

$$= 10 \times 0{\cdot}6018 = \mathbf{6{\cdot}02 \text{ ft.}}$$

$$\text{OB} = \text{AB} \times \frac{\text{OB}}{\text{AB}} = 10 \sin 53° \text{ ft.}$$

$$= 10 \times 0{\cdot}7986 = \mathbf{7{\cdot}99 \text{ ft.}}$$

EXERCISE II. a.

1. Draw (on squared paper) a quadrant of a circle of convenient radius and use it to find the values of sin 25°, cos 25°, sin 35°, cos 35°, sin 65°, cos 65°. What are the values of sin 90°, cos 90°, sin 0°, cos 0°?

2. Use Tables to write down the sines of the following: (i) 17°; (ii) 43°; (iii) 64°; (iv) 88°; (v) 23° 30′; (vi) 23° 36′; (vii) 23° 31′; (viii) 23° 35′; (ix) 38° 21′; (x) 64° 11′; (xi) 49° 2′; (xii) 85° 14′.

3. Use Tables to write down the cosines of the following: (i) 14°; (ii) 28°; (iii) 56°; (iv) 89°; (v) 66° 36′; (vi) 66° 42′; (vii) 66° 39′; (viii) 66° 41′; (ix) 62° 40′; (x) 5° 12′; (xi) 4° 32′; (xii) 53° 10′.

4. Show that the triangles in Fig. 36 are right-angled;

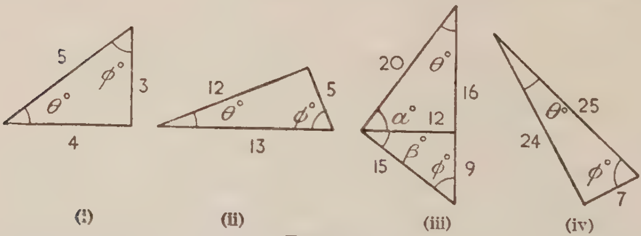


FIG. 36.

(iii) contains three such triangles. Write down the sine and cosine of each marked angle.

5. Using the data of Fig. 37, write the following as trigonometrical ratios, in more than one way if possible:

(i) $\frac{AC}{BC}$; (ii) $\frac{PQ}{PR}$; (iii) $\frac{GF}{EF}$; (iv) $\frac{XY}{YZ}$; (v) $\frac{QR}{PR}$; (vi) $\frac{AC}{AB}$;

(vii) $\frac{YZ}{XZ}$; (viii) $\frac{EG}{GF}$; (ix) $\frac{AB}{BC}$; (x) $\frac{QR}{QP}$; (xi) $\frac{EG}{EF}$; (xii) $\frac{XY}{XZ}$.

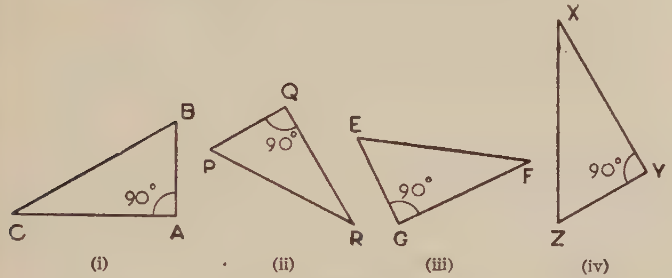


FIG. 37.

6. In Fig. 38 the triangles are right-angled, and the given side is in each case the hypotenuse. Find the other sides.

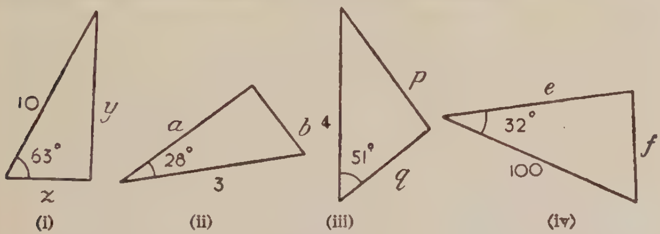


FIG. 38.

7. In Fig. 37 solve the following:

(i) $BC=3$, $\angle C=22° 34'$, find AC;
(ii) $PR=2$, $\angle R=21° 44'$, find PQ;
(iii) $EF=100$, $\angle E=71° 8'$, find FG;
(iv) $XZ=4$, $\angle X=25° 53'$, find XY;
(v) $QR=10$, $\angle P=61° 30'$, find PQ;
(vi) $BC=5$, $\angle B=70° 45'$, find AB;
(vii) $XY=3$, $\angle X=31° 24'$, find YZ;
(viii) $EF=5$, $\angle E=74° 22'$, find EG.

8. Find the sine, cosine and tangent of each marked angle in Fig. 39; note that the triangles are not right-angled.

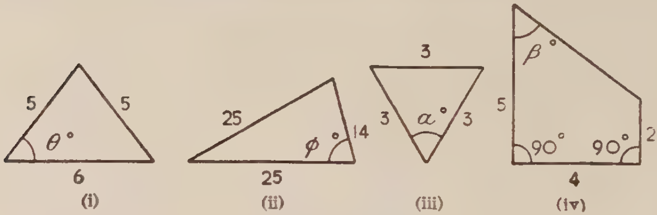


FIG. 39.

9. A hill slopes upwards at an angle of 18° with the horizontal. What height does a man rise when he walks 100 yd. up the slope?

10. B is 2000 yd. N. 34° E. from A. How much is B (i) East, (ii) North of A?

11. The string of a kite is 400 ft. long, and makes an angle of 62° with the horizontal. What is the height of the kite?

12. Find the area of the parallelogram in Fig. 40.

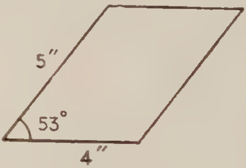


FIG. 40.

13. What are the values of cos 32° and sin 58°? Why are they equal?

14. Find a value of x if

(i) $\cos x° = \sin 48°$;
(ii) $\sin x° = \cos 47° 30'$;
(iii) $\cos x° = \sin 21° 47'$;
(iv) $\sin x° = \cos 15° 21'$?

15. Find from a Table of *sines* the value of cos 14° 27′.

16. The legs of a pair of dividers are each 12 cm. long, and are opened to an angle of 31°. Find the distance between their points.

17. Repeat No. 16, if the angle is 170°.

18. Fig. 41 represents a semicircle; find the length (i) of the chord, (ii) of the portion of the tangent it cuts off.

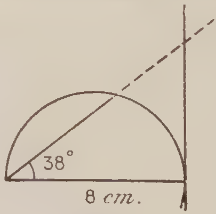


FIG. 41.

19. The vertical angle of a cone is 23° and the length of a slant edge is 2·5 in. What is the diameter of the base?

20. A telegraph pole AB, 18 ft. high, is stayed by a tie CD, 12 ft. long, making 67° with the horizontal. How far is the point of attachment C from A?

21. A regular pentagon is inscribed in a circle of radius 5 cm. What is the length of its side?

22. A ladder 20 ft. long leans against the side of a house. What distance must the foot of the ladder be pushed to increase the angle of slope of the ladder from 60° to 65°?

23. The diagonals of a rectangle are 12 cm. long and contain an angle of 17° 30′. Find its length and breadth.

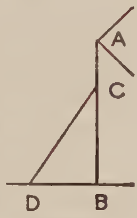


FIG. 42.

24. Find the projection CD of AB on the ground line HK.

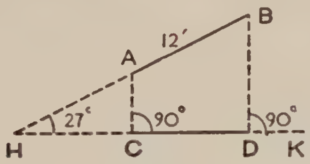


FIG. 43.

25. A straight path is inclined at 4° to the horizontal. What is the distance along the path between the points where the 100 ft. and 200 ft. contours are crossed?

Construction of acute angles of given sine or cosine.

We have seen that the sine and cosine of an acute angle can have any value between 0 and 1. By using the tables we can find approximately the size of the angle if the sine or cosine is given.

Example III. Find θ, given that

$$\text{(i) } \sin\theta^\circ = 0{\cdot}86,$$

$$\text{(ii) } \cos\theta^\circ = 0{\cdot}74.$$

(i) From the tables, $\sin 59^\circ\, 18' = 0{\cdot}8599.$

Difference for $1' = 0{\cdot}0001$;

$\therefore \sin 59^\circ\, 19' \simeq 0{\cdot}8600.$

(ii) From the tables, $\cos 42° 18' = 0{\cdot}7396$.

Difference for $2' = 0{\cdot}0004$;

$\therefore \cos 42° 16' \simeq 0{\cdot}7400$.

Note that the $2'$ is *subtracted* because the angle decreases if its cosine is increased.

Definition. The angle whose sine is x is often written $\sin^{-1}(x)$; the angle whose cosine is x is written $\cos^{-1}(x)$, and the angle whose tangent is x is written $\tan^{-1}(x)$; or, more shortly $\sin^{-1}x$, $\cos^{-1}x$ and $\tan^{-1}x$.

Thus $\sin^{-1} 0{\cdot}86 \simeq 59° 19'$ and $\cos^{-1} 0{\cdot}74 \simeq 42° 16'$.

Example IV. Construct the angle whose sine is equal to 0·77.

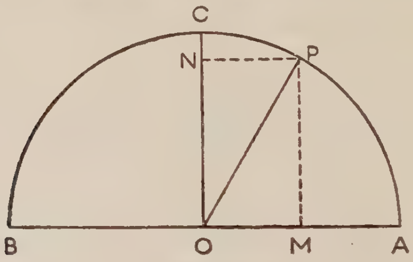


FIG. 44.

Draw a circle of unit radius, centre O, diameter AOB.

(*Note.* The reader should draw his own figure, preferably on squared paper, and take 1 dm. or 5 inches as his unit. Fig. 44 represents part of a circle of radius 1 inch. It is unnecessary to draw more than a quarter of the circle.)

Draw the radius OC at right angles to OA, and cut off ON equal to 0·77 units: the parallel through N to OA cuts the circle at P; PM is drawn perpendicular to OA.

Then $$\sin \text{AOP} = \frac{\text{MP}}{\text{OP}} = \frac{\text{ON}}{\text{OP}} = \frac{0{\cdot}77}{1} = 0{\cdot}77.$$

By measurement we find $\angle \text{AOP} \simeq 50{\cdot}5°$.

From the Tables we see that $\angle \text{AOP} \simeq 50° 21'$.

Note. (i) To construct (say) $\cos^{-1}(0{\cdot}65)$, cut off a length OM from OA, such that OM $= 0{\cdot}65$ units, and draw MP perpendicular to OA, cutting the circle at P, then $\angle$ AOP is the required angle.

(ii) The use of squared paper is advised, merely because it saves time.

EXERCISE II. b.

1. Find, by drawing and measurement,

(i) $\sin^{-1}(0{\cdot}4)$; (ii) $\sin^{-1}(0{\cdot}7)$; (iii) $\sin^{-1}(\frac{3}{4})$; (iv) $\sin^{-1}(0{\cdot}92)$.

[Use squared paper.]

2. Find, by drawing and measurement,

(i) $\cos^{-1}(0{\cdot}31)$; (ii) $\cos^{-1}(0{\cdot}63)$; (iii) $\cos^{-1}(0{\cdot}81)$; (iv) $\cos^{-1}(0{\cdot}91)$.

[Use squared paper.]

3. Use Tables to write down the angles whose sines are:

(i) 0·3907; (ii) 0·9613; (iii) 0·7694; (iv) 0·4493; (v) 0·4509; (vi) 0·4498; (vii) 0·4504; (viii) 0·2345; (ix) 0·3199; (x) 0·9648.

4. Use Tables to write down the angles whose cosines are:

(i) 0·5592; (ii) 0·7880; (iii) 0·8712; (iv) 0·1805; (v) 0·1788; (vi) 0·1794; (vii) 0·1802; (viii) 0·7585; (ix) 0·8631; (x) 0·9834.

5. Use Tables to evaluate

(i) $\sin^{-1}(0{\cdot}5265)$; (ii) $\cos^{-1}(0{\cdot}3100)$; (iii) $\tan^{-1}(0{\cdot}6308)$; (iv) $\cos^{-1}(0{\cdot}5203)$; (v) $\sin^{-1}(0{\cdot}0114)$; (vi) $\tan^{-1}(3{\cdot}009)$.

6. Use Tables to evaluate the marked angles in Fig. 36.

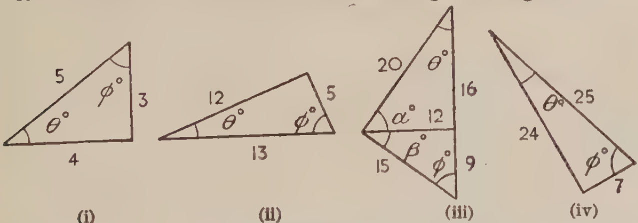


FIG. 36.

7. A ladder 12 ft. long leans against the wall, and one end is 3 ft. from the wall. What angle does the ladder make with the wall?

8. What is the angle of slope of a road if a man has risen 30 ft. vertically after walking 100 yards up the road?

9. Use Tables to evaluate the marked angles in Fig. 39.

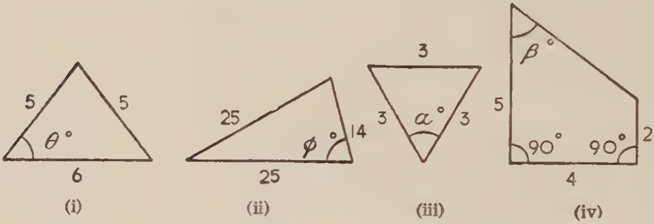


FIG. 39.

10. The sides of a parallelogram are 4, 5 inches and its area is 12 sq. in. Calculate its angles.

11. Taking both possible meanings of the term "gradient" (see p. 25) find the inclination to the horizontal of roads whose gradients are (i) 1 in 3, (ii) 1 in 10, (iii) 1 in 30, (iv) 1 in 100. Which meaning gives the greater inclination, and why?

12. A road 800 yards long is represented on a map, scale 1 : 20,000, by a line of length 1·42 inches. What is the average inclination of the road to the horizontal?

13. A pencil 6″ long casts a shadow 5″ long when the sun is vertically overhead. What is the inclination of the pencil to the horizontal?

14. The pole of a bell-tent is 8 ft. high, and the length of the slant side is 11 ft. What angle does the side make with the ground?

15. A soldier's legs are 32″ long. At what angle are they inclined when he stands at ease with his feet 10″ apart? What difference does this make to his height?

16. In Fig. 45, calculate ∠ BAC.

17. The legs of a pair of dividers are each 12 cm. long and are opened so that the points are 5 cm. apart. What is the angle between the legs?

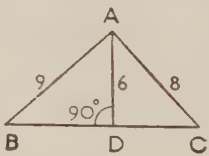


FIG. 45.

18. With the data of No. 17, the points rest on the surface of a sphere of radius 8 cm. What angle do they subtend at the centre of the sphere?

19. In Fig. 46, calculate $\angle$ CAD.

20. The tops of two vertical poles of heights 20, 15 ft. are joined by a taut wire 12 ft. long. What is the angle of slope of the wire ?

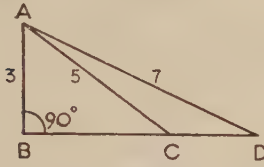


FIG. 46.

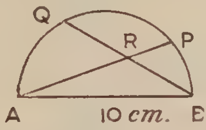


FIG. 47.

21. In Fig. 47, AB is a diameter; AP = 8·5 cm., BQ = 7·5 cm. Calculate (i) $\angle$ PAB, (ii) $\angle$ PRQ.

22. The centres of two circles of radii 7, 3 cm. are 12 cm. apart. Calculate the angle between their exterior common tangents PQ, RS.

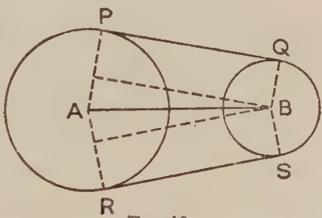


FIG. 48.

23. With the data of No. 22, calculate the angle between the interior common tangents.

24. A uniform sphere of radius 6 cm. is suspended by a string AB 9 cm. long from a point A in a smooth vertical wall AD. Calculate $\angle$ BAD. [Statical considerations show that AB produced passes through the centre of the sphere.]

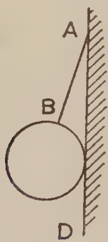


FIG. 49.

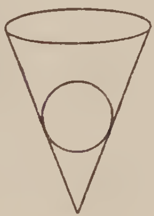


FIG. 50.

25. A sphere of radius 8 cm. rests inside a conical funnel whose axis is vertical; the highest point of the sphere is 22 cm. above the vertex of the cone. Find the angle of the cone.

26. The diameter of a cylindrical roller is 30 inches and a handle OA, 5 ft. long, is attached to its axis O, about which it can rotate. If the roller is stationary on level ground, find the greatest angle through which the handle can swing. [See Fig. 51.]

27. Find a value of θ if

(i) $\sin \theta^\circ = 2 \sin \phi^\circ$ and $\phi^\circ = 23^\circ$;

(ii) $\cos \theta^\circ = 2 \cos \phi^\circ$ and $\phi^\circ = 72^\circ$.

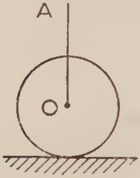


FIG. 51.

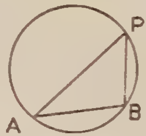


FIG. 52.

28. A chord AB of a circle of radius 6 cm. is 5 cm. long. Calculate $\angle$APB.

29. With the data of No. 28, if

$\angle ABP = 105^\circ$,

calculate AP.

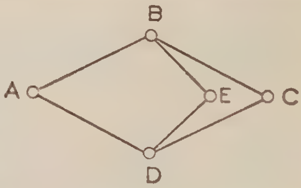


FIG. 53.

30. A mechanism (Peaucellier's cell) consists of four equal rods AB, BC, CD, DA, each of length 10 in., and two other equal rods BE, ED, each of length 7 in., smoothly jointed as shown. If $\angle BAD = 32^\circ$, calculate $\angle$BED. Find also the maximum angle between AB and AD.

31. In Fig. 54, calculate the length of the perpendicular from B to AC.

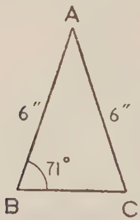


FIG. 54.

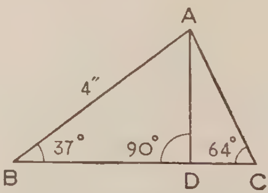


FIG. 55.

32. Calculate in Fig. 55 the lengths of AD, BD, BC.

The following exercise may be reserved for a second reading.

EXERCISE II. c.

1. What is the distance of a place in latitude 53° N. from the axis of the Earth ?

(Radius of Earth = 4000 miles.)

2. Taking the length of the Equator as 25,000 miles, find the length of the parallel of latitude in latitude 53° N.

3. A man walks 100 yd. up a slope of 22° and then 50 yd. up a slope of 18°. How far is he (i) vertically, (ii) horizontally, from his starting point ?

4. AE, BF are vertical standards ; find the heights of C, D above the ground line EF and the length of AB. (Fig. 57.)

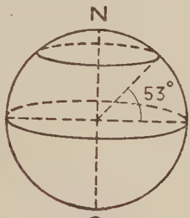


FIG. 56.

5. A mining gallery descends for 100 yd. at an angle of 13° to the horizontal and then for 200 yd. at an angle of 7° to the horizontal. How far is the point reached below the level of the starting point ?

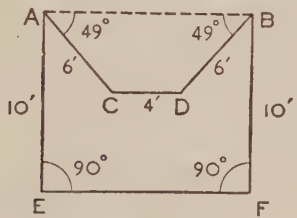


FIG. 57.

6. A man starts at O and walks 1 mile N. 21° W. to A, then 2 miles N. 43° E. to B. How far (i) North, (ii) East is B from O ?

7. The tip of a pendulum 3 ft. long rises 6 inches above its lowest point in each swing. Find the angle of swing.

8. A man starts at O and walks 1 mile N. 27° E. to A, then turns to his right through 90° and walks half a mile to B. What is the bearing of B from O ?

9. A man wishes to row straight across a stream which is running at 1 mile per hour ; he can row at $2\frac{1}{2}$ m.p.h. through the water. At what angle to the line of the stream must he point his boat ?

10. A pendulum 5 ft. long swings through an angle of 12° on each side of the vertical. How high does its tip rise above its lowest point ?

11. ABC represents the path of a bullet fired from A at an angle of 5° to the horizontal AE. If C is its position after t seconds, $AD = 2000t$ feet and $DC = 16t^2$ feet. Find the height of the bullet after (i) 1 sec., (ii) t sec. When will it hit the ground ?

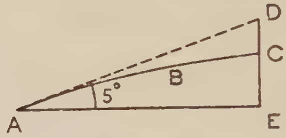


FIG. 58.

12. A rectangular block leans against a wall at B with the corner A on the ground. Find the height of C above the ground. [Fig. 59.]

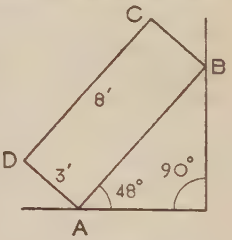


FIG. 59.

13. A regular pentagon ABCDE is inscribed in a circle of radius 4 inches. Find the length of the perpendicular (i) from A to CD, (ii) from B to AC.

14. Taking a degree of longitude at the Equator as 69 miles, find the latitude of a place where a degree of longitude is 30 miles.

15. A wheel of radius 2 ft. rests at B against an obstacle 6 in. high as shown ; the wheel is then pushed on to the top of the obstacle, which is level, turning about B. Through what angle does each spoke of the wheel turn ?

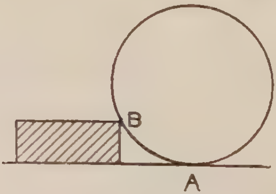


FIG. 60.

16. ABC is a triangle, right-angled at C ; CN is the perpendicular from C to AB. Write down cos A in two different forms, and hence prove that $AC^2 = AN \, . \, AB$. Similarly prove that $BC^2 = BN \, . \, BA$.

What is the connection between these results and the usual proof and enunciation of Pythagoras' theorem ?

17. A, B are two billiard balls at distances 20, 30 in. from a perfectly elastic cushion CD. The ball A is struck along AP and hits

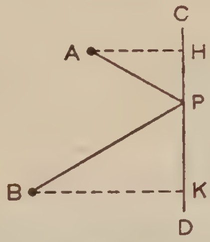


FIG. 61.

B full on the rebound after travelling altogether 70 in. ; neglecting the size of the balls, and assuming that AP, PB make equal angles with CD, calculate $\angle$APB.

18. A boat sailing against the wind from X to a place Y due East of X takes a course either N. 64° E. or S. 64° E. alternately. What is the distance of Y from X, if the boat has to travel 5000 yards ?

FIG. 62.

19. What can you say about a triangle ABC in which $\cos A = \sin B$, if B is acute ?

20. One solution of the equation $\sin x° + \cos x° = 1{\cdot}292$ is $x = 21$. Find another solution.

21. Shew that in any triangle ABC,

$$\text{(i) } \sin\frac{A}{2} = \cos\frac{B+C}{2}, \quad \text{(ii) } \cos\frac{B}{2} = \sin\frac{A+C}{2}.$$

22. Write down an equation that may connect x and y if

$$\cos x° = \sin y°.$$

Find a value of x if $\cos x° = \sin 2x°$.

23. Find a value of x if

$$\text{(i) } \cos x° = \sin (x+20)°, \quad \text{(ii) } \sin x° = \cos (x-16)°.$$

24. Fig. 63 represents a section of a rectangular box with its lid DE; a sphere of diameter 16″ is placed in the box. What is the least angle DE makes with BC?

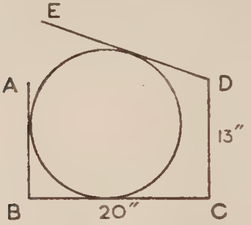


FIG. 63.

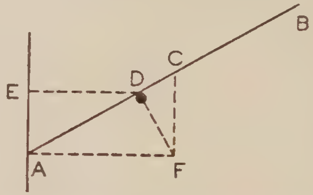


FIG. 64.

25. A uniform rod AB, mid-point C, rests on a smooth peg at D with its end A against a smooth vertical wall. It can be proved by statical principles that the horizontal line through A cuts the vertical line through C at a point F such that FD is perpendicular to AD. If AB = 27 in., and if the peg is 4 inches from the wall, find the angle AB makes with the vertical.

26. Fig. 65 represents the lid AK, pivoted at A, of a hot-water jug AKLM. AB and BC are perpendicular rods attached rigidly to each other and the lid. The upper horizontal surface DE of the handle prevents the lid opening fully by acting as a stop for C; DEF is a straight line. If AB = 1 cm., BC = 2·5 cm., AF = 0·5 cm., and $\angle$BAK = 90°, find the maximum angle through which the lid can turn.

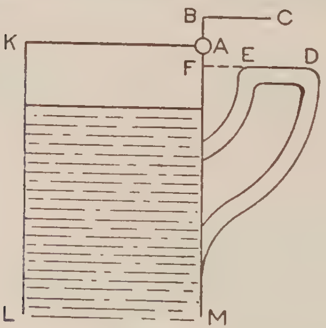


FIG. 65.

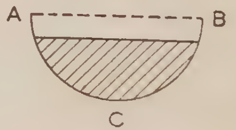


FIG. 66.

27. A trough has a semi-circular section ACB, diameter 18 inches, and contains water to a depth of 7 inches; initially AB is horizontal; through how large an angle can it be tilted before any water is upset?

CHAPTER III.

COSECANT, SECANT AND COTANGENT.

THE reciprocals of the Sine, Cosine and Tangent of an angle are called respectively the **cosecant**, **secant** and **cotangent**, and are written more shortly as cosec, sec, and cot.

Thus
$$\operatorname{cosec}\theta=\frac{1}{\sin\theta}$$
$$\sec\theta=\frac{1}{\cos\theta},$$
$$\cot\theta=\frac{1}{\tan\theta}.$$

Since the sine and cosine of an angle cannot be greater than 1, *the cosecant and secant of an angle cannot be less than* 1.

Further, since $\sin\theta$ and $\tan\theta$ increase as θ increases, it follows that *cosec θ and cot θ decrease as θ increases*; but since $\cos\theta$ decreases as θ increases it follows that *sec θ increases as θ increases*. We see, therefore, that the **co**sine, **co**secant, **co**tangent of an angle all decrease when the angle increases; the common prefix **co** makes this easy to remember. Consequently, when four-figure Tables are used, the difference columns must be *subtracted* for an increase of θ in the $\cos\theta$, $\operatorname{cosec}\theta$ $\cot\theta$ Tables.

Complementary angles.

By definition, with the notation of Fig. 67, we have

$$\operatorname{cosec}\theta^\circ=\frac{z}{x};\qquad \sec\theta^\circ=\frac{z}{y};$$

and
$$\operatorname{cosec}(90^\circ-\theta^\circ)=\frac{z}{y};\qquad \sec(90^\circ-\theta^\circ)=\frac{z}{x};$$

$$\therefore\ \operatorname{cosec}\theta^\circ=\sec(90^\circ-\theta^\circ)\quad\text{and}\quad \sec\theta^\circ=\operatorname{cosec}(90^\circ-\theta^\circ).$$

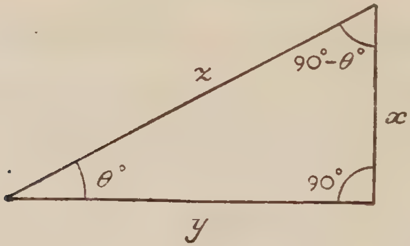


FIG. 67.

Hence the cosecant of any angle equals the secant of its complement and vice-versa.

Further,
$$\cot\theta^\circ=\frac{y}{x}=\tan(90^\circ-\theta^\circ),$$

and
$$\cot(90^\circ-\theta^\circ)=\frac{x}{y}=\tan\theta^\circ.$$

$\therefore$ *the cotangent of any angle equals the tangent of its complement and vice-versa.*

In fact, *all trigonometrical ratios are equal to the* co-*ratio of the complementary angle and vice-versa.*

One advantage of having all six Trigonometrical ratios defined and tabulated is that numerical work and statements of Trigonometrical facts and formulae can be simplified by using the most suitable ratios. This is illustrated in the following examples.

Example I. In the given triangle, find the length of AC.

$$\mathrm{AC}=\frac{\mathrm{AC}}{10}\times 10=10 \operatorname{cosec} 55^\circ$$
$$=10\times 1{\cdot}2208$$
$$\simeq \mathbf{12{\cdot}2\ cm.}$$

Note. This method is simpler than saying

$$\frac{10}{\mathrm{AC}}=\sin 55^\circ;$$

$$\therefore\ \mathrm{AC}=\frac{10}{\sin 55^\circ}=\frac{10}{0{\cdot}8192}, \text{ etc.}$$

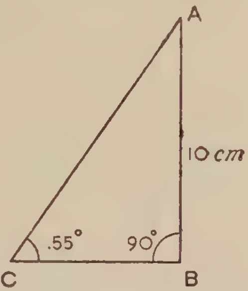


FIG. 68.

Example II. In the given triangle, find the angle A.

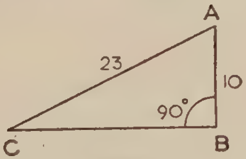


FIG. 69.

$$\sec \mathrm{A}=\tfrac{23}{10}=2{\cdot}3;$$
$$\therefore\ \mathrm{A}=\mathbf{64^\circ\ 14'}.$$

Note. This method makes the calculation simpler than saying $\cos \mathrm{A}=\frac{10}{23}$, etc.

EXERCISE III. a.

1. Use Tables to write down the values of the following:

(i) cosec 41°; (ii) cosec 41° 36′; (iii) cosec 41° 38′
(iv) cosec 75° 22′; (v) sec 28°; (vi) sec 28° 18′;
(vii) sec 28° 22′; (viii) sec 70° 43′; (ix) cot 44°;
(x) cot 45° 18′; (xi) cot 45° 20′; (xii) cot 83° 10′.

2. Use Tables to find the following angles:

(i) $\operatorname{cosec}^{-1}(1{\cdot}1992)$; (ii) $\operatorname{cosec}^{-1}(1{\cdot}2001)$; (iii) $\operatorname{cosec}^{-1}(2{\cdot}4053)$;
(iv) $\sec^{-1}(1{\cdot}3996)$; (v) $\sec^{-1}(1{\cdot}4102)$; (vi) $\sec^{-1}(2{\cdot}2542)$;
(vii) $\cot^{-1}(0{\cdot}6694)$; (viii) $\cot^{-1}(0{\cdot}6707)$; (ix) $\cot^{-1}(1{\cdot}5920)$.

3. Write down the cosecant, secant and cotangent of each of the marked angles in Fig. 36.

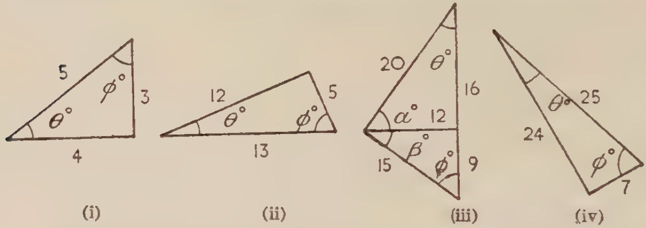


FIG. 36.

4. Using the data of Fig. 37, write the following as Trigonometrical ratios in two ways.

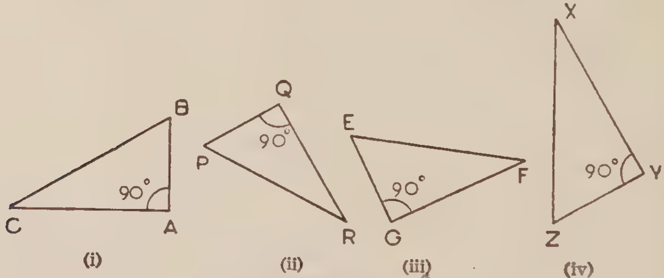


FIG. 37.

(i) $\frac{BC}{AB}$; (ii) $\frac{PR}{QR}$; (iii) $\frac{EG}{GF}$; (iv) $\frac{XZ}{YZ}$;
(v) $\frac{AC}{AB}$; (vi) $\frac{PR}{PQ}$; (vii) $\frac{EF}{EG}$; (viii) $\frac{YZ}{XY}$;
(ix) $\frac{AB}{BC}$; (x) $\frac{PQ}{QR}$; (xi) $\frac{EF}{GF}$; (xii) $\frac{XZ}{XY}$;
(xiii) $\frac{QR}{PR}$; (xiv) $\frac{BC}{AC}$; (xv) $\frac{PQ}{PR}$.

5. Using the data and notation of Fig. 37, write down simple expressions for the following:

(i) sec C; (ii) cot P; (iii) cosec F; (iv) tan X;
(v) cot B; (vi) sec R; (vii) cot E; (viii) sec Z;
(ix) cosec B; (x) sin R; (xi) AB sec ABC;
(xii) QR cosec QPR; (xiii) FG cot GEF.

6. Evaluate as shortly as possible:

(i) $\dfrac{1}{\sin 20^\circ}$; (ii) $\dfrac{1}{\sec 52^\circ}$; (iii) $\dfrac{1}{\tan 39^\circ}$;

(iv) $\dfrac{1}{\operatorname{cosec} 61^\circ}$; (v) $\dfrac{\sec 40^\circ}{\operatorname{cosec} 50^\circ}$; (vi) $\dfrac{\tan 15^\circ}{\cot 75^\circ}$;

(vii) $\tan 40^\circ . \tan 50^\circ$; (viii) $\cos 35 . \operatorname{cosec} 55^\circ$.

7. Find the marked angles in the triangles in Fig. 70.

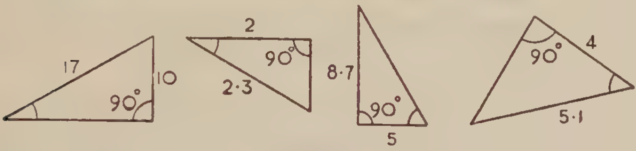


FIG. 70.

8. Find the remaining sides in the triangles in Fig. 71.

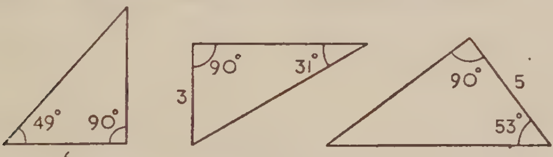


FIG. 71.

9. Find from the Tables the values of:
(i) $\cos 48^\circ 30'$ and $\sin 41^\circ 30'$; (ii) $\tan 16^\circ 25'$ and $\cot 73^\circ 35'$; (iii) $\operatorname{cosec} 37^\circ 10'$ and $\sec 52^\circ 50'$.

10. Find a value of x if

(i) $\cos x^\circ = \sin 62^\circ$; (ii) $\tan x^\circ = \cot 14^\circ$;

(iii) $\sin x^\circ = \cos 51^\circ 25'$; (iv) $\sec x^\circ = \operatorname{cosec} 15^\circ 42'$;

(v) $\cot x^\circ = \tan 19^\circ 47'$; (vi) $\operatorname{cosec} x^\circ = \sec 71^\circ 10'$.

11. Find from a Table of *secants* the value of cosec 64° 17′.

12. In Fig. 72, $\angle BAC = 90^\circ = \angle ADB$; write the following in terms of the lengths in the figure:

(i) sec ABC;
(ii) cot ACB;
(iii) cosec BAD;
(iv) tan BAD = cot DAC;
(v) cosec DAC = cosec ABC;
(vi) sec BAD = sec ACB.

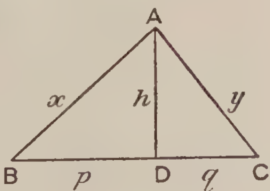


FIG. 72.

13. A man on the top of a tower 200 ft. high measures the angle of depression of a milestone as 24°. How far is the milestone from the man ?

14. A chord of length 8 cm. subtends an angle of 100° at the centre of the circle ; what is the radius ?

15. A kite is flying at a height of 300 ft. above the ground at the end of a string which makes 33° with the vertical. What is the length of the string ?

16. A quay-side stairway descends from the quay at an angle of 17° with the horizontal : the height of the quay is 26 ft. What is the length of the stairway ?

17. Each leg of a tripod is 5 ft. long and makes an angle of 57° with the ground. What is the height of the apex of the tripod ?

18. A buoy is attached to the bed of a channel by a chain which makes with the vertical an angle of 14° when the flow of the tide has reduced the depth of water to 18 ft. What is the length of the chain ?

19. A boat P is 4 sea-miles due west of a lighthouse Q, and is steaming at 15 knots on a course N. 57° E. After what time will P be due North of Q ?

20. The tangents from a point A to a circle are 3·5 in. long and contain an angle of 97° ; find the distance of A from the centre.

21. A flagstaff snaps at a point P, 8 ft. above its base A, and the top PB rests at an angle of 17° with the ground. Find the original height of the flagstaff.

FIG. 73.

22. A portion of road AB which slopes uphill at an angle of 7° is represented on a map of scale 4 inches to the mile by a line of length 3·3 inches. Find the length of AB in yards.

23. A taut elastic string joins two points A, B 30 inches apart and at the same level ; when a body is attached to the mid-point C of the string, AC and BC make angles of 8° 20′ with the horizontal. How much has the string stretched ?

24. One angle of a rhombus is 37°, and the shorter diagonal is 6 cm. Find the length of a side.

25. The diagonals of a rectangle intersect at an angle of 33° 48′ and the length of one side is 5 inches. What is the length of a diagonal ? [Two possible answers.]

26. In Fig. 72 of No. 12, AD = 6 cm., $\angle ADB = 90°$, $\angle ABC = 32°$, $\angle ACB = 71°$. Calculate AB, AC, BC.

27. A pendulum OA, 6 ft. long, is suspended from O; only that portion of it is visible which is above a horizontal line BC, 4 ft. below O. How much more of the pendulum is visible when it is at an angle of 20° with the vertical than when at an angle of 10°?

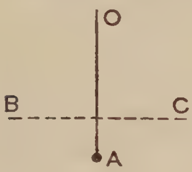


FIG. 74.

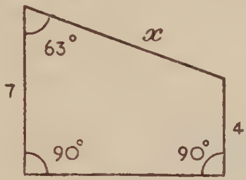


FIG. 75.

28. In Fig. 75, find x.

29. A sphere of radius 6 cm. rests inside a hollow cone of vertical angle 64°, base-radius 10 cm., with axis vertical and apex downwards. Find the distance of the centre of the sphere from the base of the cone.

30. In Fig. 76, $\angle ABC = 73° = \angle ACB$. Find the diameter of the circle,

(i) if $AB = 6$ cm.,

(ii) if $BC = 6$ cm.

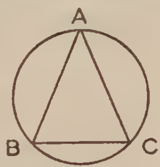


FIG. 76.

The following Exercise may be reserved for a second reading.

EXERCISE III. b.

1. EF is a fence 5 ft. high at a distance of 3 ft. from the wall OD of a house. What is the length of a ladder AB inclined at 72° to the horizontal just grazing the fence?

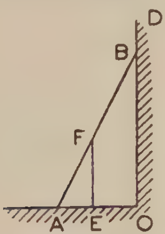


FIG. 77.

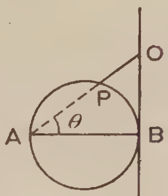


FIG. 78.

2. In Fig. 78, the diameter AB is d inches long and BO is a tangent; prove that PO is $d(\sec\theta - \cos\theta)$ inches.

3. From the top of a cliff h feet high the angles of depression of two boats in the same vertical plane as the observer are $\theta°$ and $\phi°$. $\theta > \phi$. Express the distance between the boats in terms of h, θ, ϕ.

4. In Fig. 79, ON $= c$, NA $= d$; prove that PQ $= d \operatorname{cosec} \theta - c \sec \theta$.

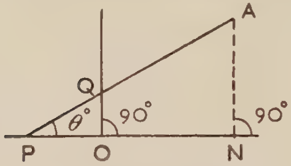


Fig. 79.

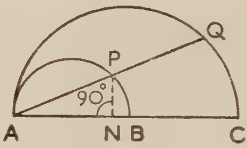


Fig. 80.

5. Fig. 80 represents two semi-circles: $\angle$PAB $= 35°$, PQ $= 4$ cm. Calculate the length of BC. If also PN $= 3$ cm., calculate the length of AC.

6. PN is perpendicular to the diameter AB; AP $= a$. Find NB in terms of a, θ. [Fig. 81.]

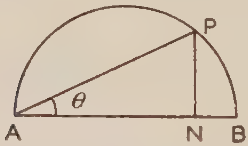


Fig. 81.

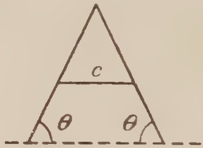


Fig. 82.

7. The cord of a pair of steps is c ft. long and, when taut, is h ft. above the ground. Find the length of each arm of the steps in terms of h, c, θ. [Fig. 82.]

8. In Fig. 83, ABCD is a rectangle; PQ $= a$. Find AC, PC in terms of a, θ.

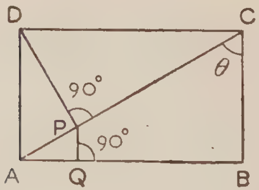


Fig. 83.

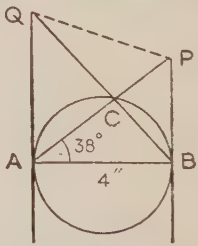


Fig. 84.

9. In Fig. 84, AB is a diameter and AQ, BP are tangents. Calculate AP, BQ and $\angle$AQP.

10. In Fig. 85, the triangle ABC is inscribed in the rectangle APQR. Calculate the sides and angles of $\triangle ABC$.

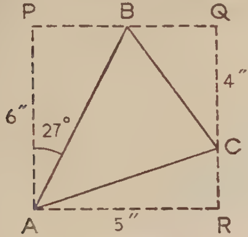


FIG. 85.

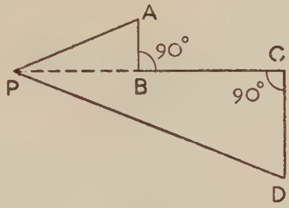


FIG. 86.

11. In Fig. 86, PBC bisects $\angle APD$; $BC=100$ yd., $\angle APD=43°$. Find how much further P is from D than from A.

12. In Fig. 87, $\angle ABC=43°$, $\angle ACB=67°$, and the radius of the circle is 10 inches. Find BC.

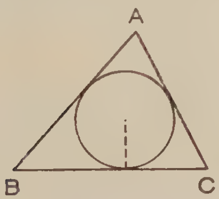


FIG. 87.

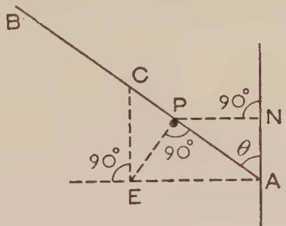


FIG. 88.

13. With the data of No. 12, find AB.

14. In Fig. 88, $PN=d$, C is the mid-point of AB. Express the length of AB in terms of d, θ.

15. In Fig. 89, $OM=p$, $ON=x$. Express PN in terms of x, p, θ.

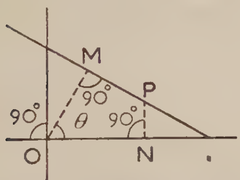


FIG. 89.

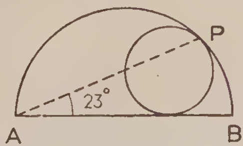


FIG. 90.

16. Fig. 90 represents a circle inscribed in a semicircle; $AP=10$ cm. Calculate the diameter of each circle.

17. Using only a table of *tangents*, find the value of cot 72° 15′.

18. What do you know about a triangle ABC, if tan A = cot B ?

19. What do you know about a triangle ABC, if sec B = cosec C, and if C is acute ?

20. One solution of the equation $\sec x° + \operatorname{cosec} x° = 3{\cdot}325$ is $x = 27$. Find another solution.

21. One solution of the equation $\tan x° + \cot x° = 2{\cdot}989$ is $x = 69$, find another solution.

22. Write down a relation which may connect x and y, if

$$\operatorname{cosec} x° = \sec y°.$$

Find a value of x if $\operatorname{cosec} x° = \sec 4x°$.

23. Find a value of x if $\tan 3x° = \cot 2x°$.

24. Find a value of θ if $\tan \theta° = \cot(\theta + 20°)$.

25. Prove that in any triangle ABC,

$$\text{(i)} \tan\frac{B+C}{2} = \cot\frac{A}{2}, \qquad \text{(ii)} \sec\frac{A+B}{2} = \operatorname{cosec}\frac{C}{2}.$$

REVISION PAPERS. R. 1-6.

R. 1.

1. Find by drawing the values of tan 16°, tan 32°, tan 64°. Write down the values obtained from the Tables.

2. In a triangle A = 90°, B = 25° 16′, b = 10 cm. Find c.

3. The vertical angle of an isosceles triangle is 67°, and the base is 8 in. long. Find the area of the triangle.

4. Find the length of the shadow of a stick 3 ft. long when the sun is at an elevation of 52°, (i) if the stick is held vertically, (ii) if the stick is inclined so as to throw the longest shadow possible.

5. The elevation of the top of a tower is 20° to an observer on the ground. What is the elevation of a point half-way up the tower ?

R. 2.

1. Find by drawing the values of sin 16°, cos 16°, sin 32°, cos 32°. Write down the values obtained from the Tables.

2. Find the height of a kite when the string is 300 feet long, and is inclined at 34° to the horizontal.

3. Find the angles of a triangle whose sides are 6 cm., 6 cm., 5 cm.

4. The pilot of an aeroplane flying horizontally at a height of 3000 feet sees a church at an angle of depression of 65°; 12 seconds later the church is vertically below him. Find his speed in feet per second.

5. AB are the posts of a soccer goal; the ball is at P; the dimensions in Fig. 91 are in yards. Within what angle must the ball be kicked along the ground if it is to enter the goal? The diameter of the ball may be neglected.

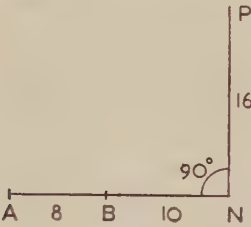


FIG. 91.

R. 3.

1. A man sitting at a window with his eye 20 ft. above the ground can just see the sun over the top of a roof 45 ft. high, which is 30 yds. from him horizontally. Find the elevation of the sun.

2. Two roads meet at O at an angle of 55°. A man at A wishes to reach a point B, where

$\angle ABO = 90°$, AO = 400 yards (Fig. 92).

How much distance will he save by going cross-country to B instead of by the roads AO, OB?

3. A boy draws the altitude AD of a triangle ABC. He measures its length correctly, but his answer is 1 per cent. too large. At what angle is the line he drew actually inclined to the base BC?

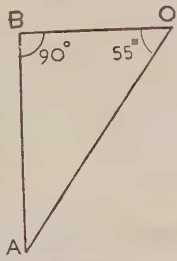


FIG. 92.

4. Evaluate as shortly as possible:

(i) $\dfrac{1}{\sin 27° 41'}$; (ii) $\dfrac{1}{\tan 39° 27'}$;

(iii) $\dfrac{1}{\operatorname{cosec} 17° 30'}$; (iv) $\cos 20° \operatorname{cosec} 70°$.

5. Find the diameter of a circle in which a chord AB, 4 cm. long, makes an angle of 35° with the diameter at A.

R. 4.

1. In a triangle ABC, $A = 42° \, 30'$, $B = 90°$, $c = 10$ cm. Find the other two sides.

2. A chord 6 inches long subtends an angle of 140° at the centre of a circle. Find the radius of the circle.

3. (i) Find a value of A° if $\sin A° = 2 \sin B°$ and $B° = 17°$.
(ii) Find a value of A° if $\operatorname{cosec} A° = 2 \operatorname{cosec} B°$ and $B° = 17°$.

4. What is the angle between the tangents to a circle of radius 6 cm. from a point 15 cm. from the centre of the circle ?

5. A track zig-zags up a steep slope from A to B ; the track is always inclined at 75° to the line AB. If $AB = 1000$ yd., what is the length of the track ?

R. 5.

1. In a triangle ABC, $A = 90°$, $a = 17 \cdot 2$ cm., $b = 10$ cm. Find B and c.

2. The centre of a golf-ball is 2 yd. from the centre of the hole, which is 3 inches in diameter. Within what angle must the ball be struck if it is to drop into the hole ?

3. A hill is said to have a gradient of 1 in 6. What is the inclination to the horizontal according to the two possible interpretations of the word gradient ?

4. A man walks 1000 yards on a bearing of 25°, and then 800 yards on a bearing of 35°. How far is he North of his starting-point ?

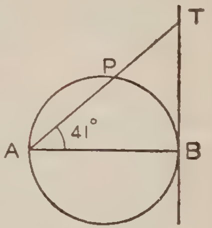


FIG. 93.

5. In Fig. 93, the diameter AB is 5 cm. long. Find the length of PT.

R. 6.

1. Find the value of $\theta + \phi$ if $\tan \theta° = 1\frac{1}{3}$ and $\tan \phi° = \frac{3}{4}$.

2. The area of the parallelogram ABCD is 10 sq. in. ; $AB = 3$ in., $BC = 4$ in., calculate $\angle ABC$.

3. In Fig. 79, p. 46, $OQ = 3$ cm., $NA = 7$ cm., $\angle OQA = 112° \, 20'$. Calculate QA and ON.

4. A regular heptagon (7 sides) is inscribed in a circle of radius 10 cm. Calculate its perimeter.

5. AB is a diameter and AC is a chord of a circle ; E is the mid-point of AC ; $AB = 10$ cm., $AC = 8$ cm. Calculate $\angle ABE$.

CHAPTER IV.

THE RIGHT-ANGLED TRIANGLE.

Ratios of special angles ; 30°, 45°, 60°.

By using Pythagoras' theorem it is easy to calculate the trigonometrical ratios of a few special angles.

Angle 45°. Draw a triangle ABC such that $CA = CB = 1$ unit of length, $\angle ACB = 90°$.

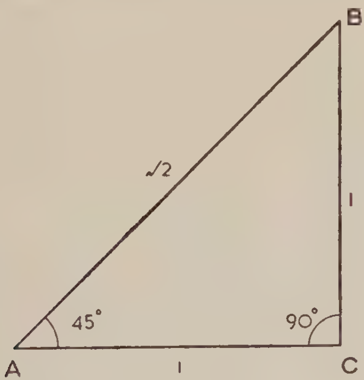


FIG. 94.

Then $\angle CAB = 45°$.

Now $AB^2 = AC^2 + CB^2 = 1^2 + 1^2 = 2$; $\therefore$ $AB = \sqrt{2}$;

$$\therefore \sin 45° = \frac{CB}{AB} = \frac{1}{\sqrt{2}}; \quad \cos 45° = \frac{AC}{AB} = \frac{1}{\sqrt{2}}; \quad \tan 45° = \frac{CB}{AC} = 1.$$

Note. $$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \simeq \frac{1 \cdot 4142}{2} = 0 \cdot 7071.$$

Angles 30°, 60°. Draw a triangle ABC such that

$$AB = BC = CA = 2 \text{ units of length,}$$

and draw BD perpendicular to AC.

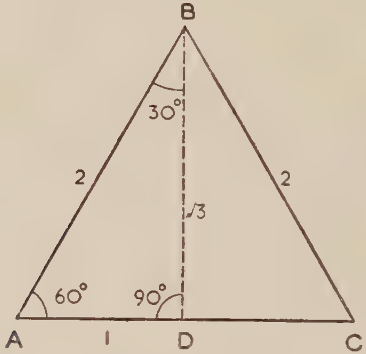


FIG. 95.

Then $\angle BAD = 60°$, $\angle ABD = 30°$; also $AD = 1$ unit of length.

Now $\quad BD^2 = AB^2 - AD^2 = 2^2 - 1^2 = 3$; $\quad \therefore BD = \sqrt{3}$;

$\therefore \sin 60° = \frac{DB}{AB} = \frac{\sqrt{3}}{2}$; $\cos 60° = \frac{AD}{AB} = \frac{1}{2}$; $\tan 60° = \frac{DB}{AD} = \sqrt{3}$,

and $\sin 30° = \frac{AD}{AB} = \frac{1}{2}$; $\cos 30° = \frac{BD}{AB} = \frac{\sqrt{3}}{2}$; $\tan 30° = \frac{AD}{BD} = \frac{1}{\sqrt{3}}$.

Note. (i) $\frac{\sqrt{3}}{2} \simeq \frac{1 \cdot 73205}{2} = 0 \cdot 8660$; $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \simeq \frac{1 \cdot 73205}{3} = 0 \cdot 5774$.

(ii) These values illustrate the relations between the ratios of complementary angles (p. 24): thus

$\sin 45° = \cos(90° - 45°) = \cos 45°$

and

$\sin 60° = \cos(90° - 60°) = \cos 30°$.

(iii) These results should be remembered; this is best done by bearing in mind the two triangles employed.

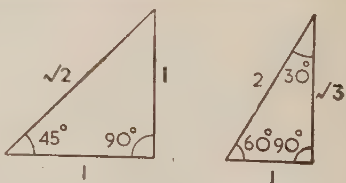


FIG. 96.

EXERCISE IV. a.

Write down the values of the following, and compare with the values given in the tables:

1. sec 45°. **2.** cosec 60°. **3.** cot 45°.

4. cosec 30°. **5.** cot 60°. **6.** cosec 45°.

7. sec 60°. **8.** cot 30°. **9.** $\dfrac{\sin 60^\circ}{\sin 30^\circ}$.

10. $\dfrac{\cos 60^\circ}{\cos 30^\circ}$. **11.** tan 30° . tan 60°. **12.** $\dfrac{\sin 45^\circ . \cos 45^\circ}{\sin 30^\circ}$.

13. The gradient of a mountain side is 1 in 1. What is its inclination to the horizontal? Is there any ambiguity in the data?

14. A climber rises 1 yard vertically for every 2 yards he climbs. What is the inclination of his path to the horizontal?

15. In Fig. 97, MN, the projection of PQ on AB, equals $\frac{1}{2}$PQ; what is the inclination of PQ to AB?

16. The shortest side of a 60° set-square is 3 inches. What are the lengths of the other sides?

17. Find the area of an equilateral triangle whose base is 8 cm.

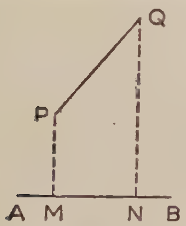


FIG. 97.

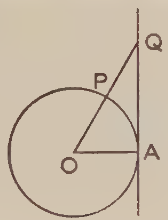


FIG. 98.

18. A ladder is 20 ft. long. How high up a vertical wall will it reach when its inclination to the horizontal is (i) 60°; (ii) 45°; (iii) 30°?

19. In Fig. 98, O is the centre and AQ is a tangent, ∠AOP = 60°; prove OP = PQ.

20. How does the length of shadow of a telegraph pole alter when the sun's elevation decreases from 60° to 30°?

21. An aeroplane flying horizontally passes vertically above a man's head: ten seconds later he notes that its elevation is 60°. When will it be 30°?

22. In Fig. 99, calculate CD, CE, $\angle$CAD; and prove

$$\sin 15° = \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

23. Use Fig. 99 to calculate cos 15°.*

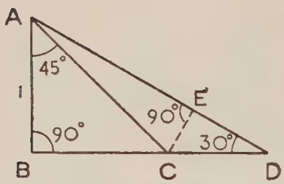


FIG. 99.

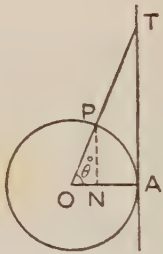


FIG. 100.

24. Fig. 100 represents a circle, centre O, radius 1 inch; $\angle$PNO = 90°; AT is a tangent.

(i) Write down the lengths of NP, ON, AT in terms of θ.

(ii) To what values do these lengths tend when $\theta°$ approaches 90°? Deduce the values of sin 90°, cos 90°, tan 90°.

(iii) To what values do these lengths tend when $\theta°$ approaches 0°? Deduce the values of sin 0°, cos 0°, tan 0°.

25. ABC is an equilateral triangle; A is the centre of the arc BEC;

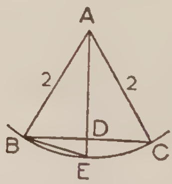


FIG. 101.

$\angle$ADB = 90°. Use the figure to prove that $\tan 15° = 2 - \sqrt{3}$.

Fundamental Formulae.

With the notation of Fig. 102, where θ is any acute angle, we have

$$\frac{\sin\theta}{\cos\theta}=\frac{x}{z}\div\frac{y}{z}=\frac{x}{z}\times\frac{z}{y}=\frac{x}{y}=\tan\theta;$$

$$\therefore\ \tan\theta=\frac{\sin\theta}{\cos\theta}.$$

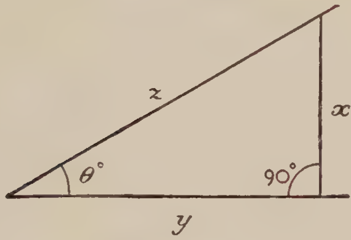


FIG. 102.

Further $$\cot\theta=\frac{1}{\tan\theta}=1\div\frac{\sin\theta}{\cos\theta}=\frac{\cos\theta}{\sin\theta};$$

$$\therefore\ \cot\theta=\frac{\cos\theta}{\sin\theta}.$$

Again, $$(\sin\theta)^2+(\cos\theta)^2=\frac{x^2}{z^2}+\frac{y^2}{z^2}=\frac{x^2+y^2}{z^2}=\frac{z^2}{z^2}=1.$$

Note. The expressions $(\sin\theta)^2$, $(\cos\theta)^2$, $(\tan\theta)^2$, *etc., are always written* $\sin^2\theta$, $\cos^2\theta$, $\tan^2\theta$, *etc.*;

$$\therefore\ \sin^2\theta+\cos^2\theta=1.$$

These results should be committed to memory.

If any one trigonometrical ratio of an angle is given it is possible to calculate the value of any other ratio of that angle without using Tables.

Example I. Given that sec $\theta = 1{\cdot}5$ calculate sin θ and cot θ.

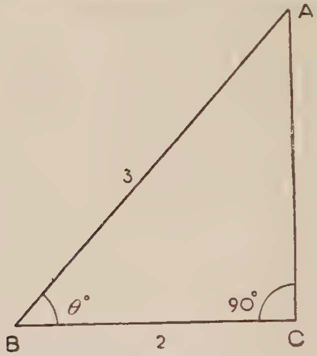


FIG. 103.

Draw the triangle ABC, so that

$$\angle \text{ACB} = 90°, \quad \text{BC} = 2, \quad \text{BA} = 3.$$

Then $\sec \text{ABC} = \frac{3}{2} = 1{\cdot}5.$

By Pythagoras, $\text{AC}^2 = 3^2 - 2^2 = 9 - 4 = 5$; $\therefore$ $\text{AC} = \sqrt{5}$;

$$\therefore \sin\theta = \sin \text{ABC} = \frac{\sqrt{5}}{3} \simeq \frac{2{\cdot}236}{3} \simeq 0{\cdot}745,$$

and

$$\cot\theta = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5} \simeq \frac{4{\cdot}472}{5} \simeq 0{\cdot}894.$$

EXERCISE IV. b.

Evaluate as shortly as possible the expressions in Nos. 1-8:

1. $\frac{\sin 50°}{\cos 50°}$. **2.** $\tan 17° \cos 17°$. **3.** $\sec 20° \sin 20°$.

4. $\cot 65° \sin 65°$. **5.** $\operatorname{cosec} 23° \cos 23°$. **6.** $\sec 71° \cot 71°$.

7. $\sin 50° \operatorname{cosec} 40°$. **8.** $\sec 15° \cos 75°$

9. If $\sin\theta = \frac{3}{5}$, calculate $\tan\theta$, $\sec\theta$.

10. If $\cot\theta = 2{\cdot}4$, calculate $\cos\theta$, $\operatorname{cosec}\theta$.

11. If $\cos\theta = \frac{3}{4}$, calculate $\sin\theta$, $\sec\theta$.

12. If $\sin\theta = 2\cos\theta$, calculate $\tan\theta$, $\operatorname{cosec}\theta$.

13. If $\sec\theta = \frac{5}{4}$, calculate $(\sin\theta + \cos\theta)^2$ and $\sin^2\theta + \cos^2\theta$.

14. Simplify (i) $\tan\theta \, . \cos\theta$; (ii) $\operatorname{cosec}\theta \, . \tan\theta$; (iii) $\frac{\cot\theta}{\operatorname{cosec}\theta}$.

15. Simplify (i) $\tan\theta \,.\, \tan(90^\circ - \theta)$; (ii) $\sin\theta \,.\, \cos(90^\circ - \theta)$; (iii) $\cos\theta \,.\, \sec(90^\circ - \theta)$.

16. Divide each side of the identity $\sin^2\theta + \cos^2\theta = 1$ by $\cos^2\theta$ and express the result in its simplest form.

17. Divide each side of the identity $\sin^2\theta + \cos^2\theta = 1$ by $\sin^2\theta$, and express the result in its simplest form.

18. Use the notation of Fig. 102 to prove that

$$1 + \tan^2\theta = \sec^2\theta.$$

19. Use the notation of Fig. 102 to prove that

$$1 + \cot^2\theta = \operatorname{cosec}^2\theta.$$

20. Given $\operatorname{cosec}\theta = p$, find (i) $\cot\theta$: (ii) $\cos\theta$ in terms of p.

21. If θ is an acute angle, prove that $\tan\theta$ is greater than $\sin\theta$.

22. Which is the greater $\cot\theta$ or $\cos\theta$, if θ is an acute angle ?

23. Which of the following equations are impossible (for real angles) ?

(i) $\sec\theta = 2$; (ii) $\operatorname{cosec}\theta = \frac{3}{4}$; (iii) $\tan\theta = 3$;
(iv) $\cos\theta = \frac{4}{5}$; (v) $\sin\theta = \frac{5}{4}$; (vi) $\cot\theta = 10$;
(vii) $\sin\theta = \sec\theta$; (viii) $\cos\theta = \frac{1}{4}\sec\theta$;
(ix) $\tan\theta = 2\sin\theta\sec\theta$; (x) $\sin\theta + \cos\theta = 2$;
(xi) $\tan\theta = \cot\theta$; (xii) $\sin^2\theta + \cos^2\theta = \frac{1}{2}$.

24. Use the fundamental formulae on p. 55 to deduce the values of $\cos 90^\circ$, $\tan 90^\circ$, given that $\sin 90^\circ = 1$.

25. Use the fundamental formulae on p. 55 to deduce the values of $\sin 0^\circ$, $\tan 0^\circ$, given that $\cos 0^\circ = 1$.

26. Evaluate $\sin^2 20^\circ + \sin^2 70^\circ$.

27. Evaluate $\cos^2 35^\circ + \cos^2 55^\circ$.

28. If $x = 2\operatorname{cosec}\theta$, $y = 3\sec\theta$, prove that $\dfrac{4}{x^2} + \dfrac{9}{y^2} = 1$.

29. The following equations occur in a dynamical problem:

$$\frac{V^2 \sin\theta\cos\theta}{32} = 150\,; \quad \frac{16V^2\sin^2\theta}{32^2} = 75\,;$$

find V and θ.

30. It is proved in Ex. IV. a., No. 22, p. 54, that

$$\sin 15^\circ = \frac{\sqrt{3} - 1}{2\sqrt{2}}\,;$$

(i) calculate the value of $\cos 15^\circ$, and prove that $\tan 15^\circ = 2 - \sqrt{3}$;
(ii) write down similar expressions for $\sin 75^\circ$, $\cos 75^\circ$, $\tan 75^\circ$.

Problems involving right-angled triangles.

If we have a *right-angled* triangle, the trigonometrical ratios enable us (i) to find either acute angle if any two sides are given, (ii) to find any side in terms of one other side, and a ratio of either acute angle.

Example II. In Fig. 104 find expressions for (i) $\angle$ACB, (ii) QR.

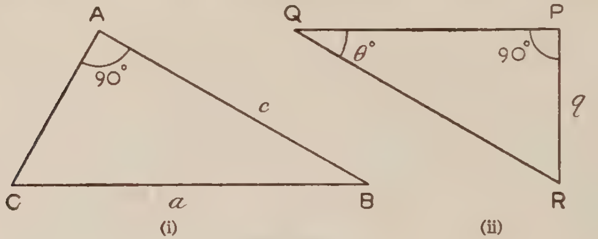


FIG. 104.

In Fig. 104 (i), $\sin \text{ACB} = \dfrac{c}{a}$; $\therefore \angle \text{ACB} = \sin^{-1}\left(\dfrac{c}{a}\right)$.

In Fig. 104 (ii), $\dfrac{\text{QR}}{q} = \operatorname{cosec}\theta$; $\therefore \text{QR} = q \operatorname{cosec}\theta$.

With a little practice, the reader should be able to write down the second step in each of these two lines without having to write down the first, and he should acquire the habit of doing so.

EXERCISE IV. c.

With the notation of Fig. 105 *write down without any preliminary working* expressions for the following: Nos. 1-12.

1. y in terms of z, θ.
2. z in terms of x, θ.
3. z in terms of y, ϕ.
4. x in terms of y, ϕ.
5. ϕ in terms of x, y.
6. θ in terms of z, x.
7. x in terms of z, θ.
8. z in terms of x, ϕ.
9. θ in terms of x, y.
10. ϕ in terms of z, x.
11. y in terms of x, θ.
12. z in terms of y, θ.

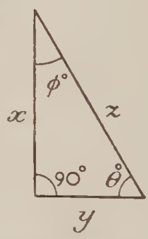


FIG. 105.

With the notation of Fig. 37 write down expressions for the following:

13. PR in terms of PQ, $\angle$R. **14.** EG in terms of GF, $\angle$E.

15. XZ in terms of ZY, $\angle$X. **16.** $\angle$R in terms of QR, PR.

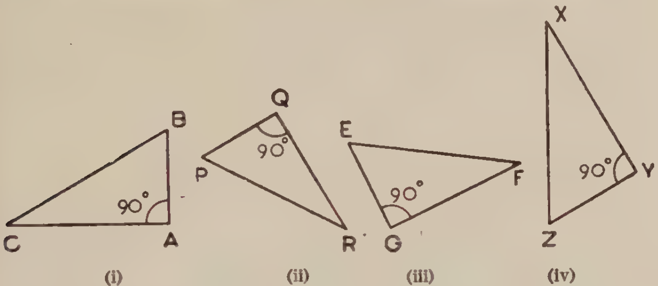


FIG. 37.

17. $\angle$X in terms of XY, YZ. **18.** QR in terms of PQ, $\angle$R.

19. XY in terms of XZ, $\angle$X. **20.** FG in terms of EF, $\angle$E.

21. A Zeppelin is 525 ft. long. What angle does its length subtend at the eye of an observer 5000 ft. vertically below its centre?

22. A captive balloon is held by a rope 400 ft. long which is inclined at 55° to the horizontal. Find the height of the balloon.

23. The eye of an observer is 5 ft. 7 in. above the ground; standing back 4 ft. 6 in. from a 7 ft. wall he can just see a distant aeroplane over the top of the wall. What is its angular elevation?

24. A skylight AB, 18 in. long, is kept open by a stick BC, 8 in. long; through what angle has AB been opened?

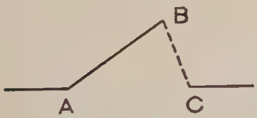


FIG. 106.

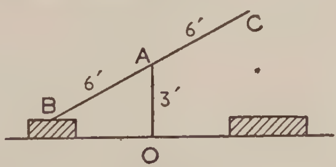


FIG. 107.

25. BAC is a see-saw, the ground blocks are each 1 ft. high; through what angle can it swing?

26. Paper ruled with parallel lines $\frac{1}{2}$-inch apart is used to divide a line AB, 4 inches long, drawn on tracing paper into 7 equal parts. What angle must AB make with the parallel lines?

27. A pendulum OA, 4 ft. long, hangs inside a case with vertical sides BC, DE and is at a distance of 18 inches from one, and 30 inches from the other. Through what angle can it swing on either side of the vertical ?

FIG. 108.

28. A fleet in line ahead is ordered to maintain a distance of 2 cables between each pair of ships; a midshipman in the bow of one ship knows that the mast of the ship in front is 150 feet from its stern and that its top is 110 ft. above his level. What angular elevation ought he to find for its top for correct stations ? (1 cable = 200 yards.)

29. At the top of a tower is a flagstaff 10 ft. high. It throws a shadow $8\frac{1}{4}$ ft. long on the ground. Find the altitude of the sun.

30. Fig. 109 shows a chair with a straight back; DE = 20 in., DB = 16 in., AC = 45 in. The chair is turned over forwards to keep the seat dry. At what angle with the ground are the legs tilted ?

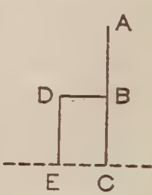


FIG. 109.

31. The sun is due South at elevation 40°; a vertical pole 9 ft. high is 4 ft. away from a vertical wall running East and West. What is the length of the shadow of the pole on the wall ?

32. Fig. 110 represents a rectangular protractor. Show that the corners lie between 29° and 30°. Find also the linear distance separating the two 50° graduations.

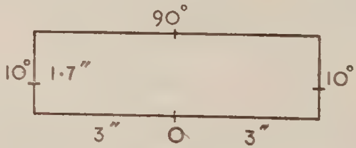


FIG. 110.

33. The letter S is formed of two semicircular arcs of radii 5, 6 cm.; from each extremity of the letter a tangent is drawn to the opposite arc. Find the acute angle between the tangents.

34. Two straight railway lines would if produced intersect at O at an angle of 140°; it is desired to connect them by a circular arc of radius 20 chains; how far from O should the rail begin to curve ?

35. In Fig. 111, ABCD is a square; calculate ∠DPC.

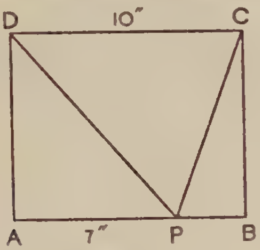


FIG. 111.

36. A straight line is drawn on a map (scale 3 inches to the mile), cutting the contour lines 600, 650, 700 ft. at A, B, C; AB=0·2 in., BC=0·3 in.; find the average angle of slope of the hills represented by AB and BC. Find also the greatest height of a vertical flagstaff at C which is invisible from A.

37. A soldier lying on the ground aims at a mark 60 ft. above the ground on a tower 300 yards away; if the bullet hits the mark it must when leaving the rifle be travelling towards a point on the tower 9 feet above the mark. Find the angle between the axis of the rifle and the line of sight.

38. O is the centre of a circle of radius 7 cm.; PM=5 cm.; ∠PMA=90°=∠QNB. Calculate QN and MN.

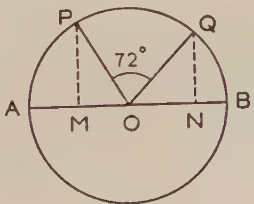


FIG. 112.

39. A marksman under cover can fire in any direction between 15° E. of N. and 9° W. of N. He is 800 yards away from a railway running East and West. What length of the track can he aim at?

40. Standing at the window of a railway carriage travelling at 48 m.p.h. along a straight track, I notice the tower of a cathedral at an angle of 50° to my left. Five minutes later it is 40° to my right. How far away is it on the second occasion?

The following Exercise may be reserved for a second reading.

EXERCISE IV. d.

1. The connecting rod AP of an engine is 6 ft. long; the crank PB of the wheel which the rod drives is 2 ft. long. Calculate the total angle through which AP oscillates.

FIG. 113.

2. Three equal rectangles are described outwards on the sides of an equilateral triangle of side 2 inches. Find their heights if their outward sides form alternate sides of a regular hexagon.

3. A cylinder, radius 5 cm., rests between a vertical wall AB and a wedge ECD. What height will the axis of the cylinder rise when the wedge is pushed up to the wall?

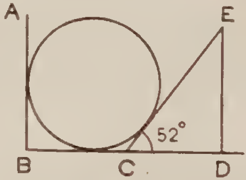


FIG. 114.

4. AB is horizontal and 8 ft. long. C can slide on the cord AB, and always remains vertically below the mid-point of AB, initially its depth is 5 ft. Calculate $\angle ACB$. If the end P is pulled down 3 ft., find the change in $\angle ACB$.

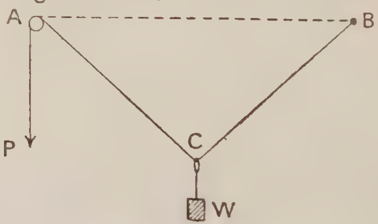


FIG. 115.

5. A boy tries to find the area of a parallelogram by multiplying together the lengths of two adjacent sides; his answer is 20 per cent. too large. Find the acute angle of the figure.

6. In $\triangle ABC$, $\angle ABC = 90°$, $AB = 5$ cm., $\angle ACB = 34°$; the bisector of $\angle BAC$ cuts BC at D. Calculate CD.

7. A garden gate ABCD, $3\frac{1}{2}$ ft. wide, is kept shut by a cord attached to B, passing over a fixed pulley F and carrying a heavy

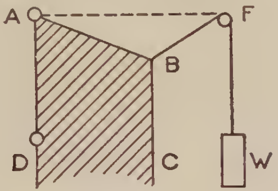


FIG. 116.

weight W. When shut, B is at F. How far does W rise when a man going through the gate opens it to an angle of 75°?

8. Fig. 117 represents in plan the equal doors of a gramophone; the hinges are at A, E, and to prevent jamming, when opening, the

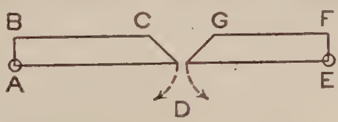


FIG. 117.

faces DC, DG are cut away as shown; $AB = \frac{5}{8}$ in., $AD = 6$ in. Find the greatest size of $\angle ADC$ compatible with clearance.

9. Fig. 118 represents a skylight BP, 18 in. long, in a roof ABCD sloping at 48° to the horizontal. BP exactly fits the opening BC, and is kept shut by a string attached to P, passing over a small pulley at C and carrying a weight W. How high does W rise when BP is opened through 20°, and what is then the vertical height of P above C?

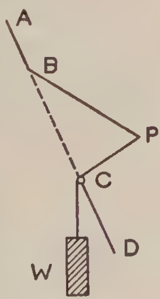


FIG. 118.

10. A man standing on a bank of a river with his eye 8 ft. above the water observes that the angle of elevation of the top of a tree on the opposite bank is 23° 45′, and the angle of depression of its image in the water is 37° 15′. What is the height of the tree top above the water and the breadth of the river?

11. In Fig. 119, $\angle PMO = 90° = \angle MQO = \angle QNO$. Express $\frac{PM}{QN}$ in terms of θ.

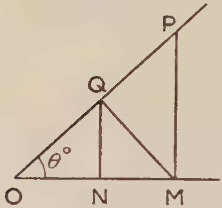


FIG. 119.

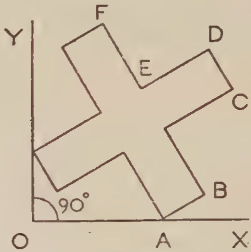


FIG. 120.

12. A symmetrical cross is tilted as shown in Fig. 120;

$$DE = EF = 2 \text{ ft.}, \quad AB = CD = 1 \text{ ft.}, \quad \angle BAX = 25°.$$

Calculate OA and the distances of D, E, F from OX and OY.

13. Fig. 121 represents a loosely fitting drawer tilted at $\theta°$ to the base. Find an equation connecting $\theta°$ with the given measurements a, b, c, d.

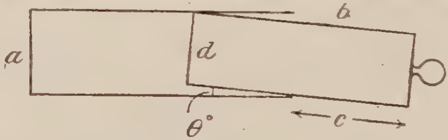


FIG. 121.

14. The vertical angle of an isosceles triangle is $2\theta°$ and the radius of its circumcircle is R inches; prove that the area of the triangle is $4R^2 \sin\theta \cos^3\theta$ sq. inches.

15. O is the centre of the rectangular top ABCD of a billiard table; a ball struck from O moves along OXY; $AB = 2a$, $BC = 2b$, $\angle OXA = \angle BXY = \theta°$. Find BY in terms of a, b, θ.

If $a = 2b$, find $\theta°$ if the ball rebounds from X into the pocket at C.

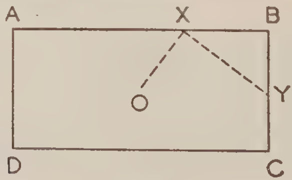


FIG. 122.

16. ABCD is the base of a rectangular tank, full of water; AB=4 ft., BC=2 ft., and the height is 1 ft. The tank is tilted about AB through an angle $\theta°$, so that some water runs out, and then returned to its original position. Prove that the new depth of water in the trough is $(1-\tan\theta°)$ feet, if $\tan\theta<\frac{1}{2}$.

17. With the data of No. 16, prove that the new depth is $\frac{1}{4}\cot\theta$, if $\tan\theta>\frac{1}{2}$. What happens if $\tan\theta=\frac{1}{2}$?

Find the value of θ if (i) one-third, (ii) two-thirds of the water overflows.

18. Fig. 123 represents a roof-frame: ADE is a circular arc with its centre on the vertical line CDO, and the tangents at A, E are parallel to CB, CF; OA=a, OB=b. Prove that

$$CD=b\tan\theta-a(\operatorname{cosec}\theta-\cot\theta).$$

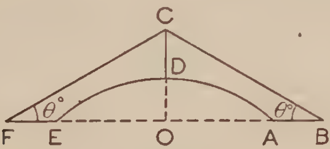


FIG. 123.

19. P is a point on a semicircle whose diameter is AB; PQ is the perpendicular from P to the tangent at B; QR is the perpendicular from Q to PB, RN is the perpendicular from R to AB; AB=d; $\angle$PAB=θ. Express QR and AN in terms of d, θ.

20. Prove that the equation $a\sin\theta°+b\cos\theta°=c$ can be solved graphically for θ, as follows: Draw two perpendicular lines OA, OB such that OA$=\frac{c}{b}$, OB$=\frac{c}{a}$; draw a circle, centre O, radius unity, and let it cut AB at P, Q; then $\theta°=\angle$AOP or $\angle$AOQ. [To prove this, draw PN perpendicular to OA.] Solve graphically $2\sin\theta+\cos\theta=2$.

21. In fig. 124, C is the mid-point of the semi-circular arc AB, centre O, radius a; $\angle$OCP$=\theta$, $\angle$QPB$=90°$; show that

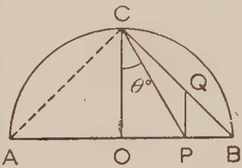


FIG. 124.

(i) $\angle$PQA$=\angle$PCA$=45°+\theta$; (ii) AP$=a(1+\tan\theta)$;

(iii) PQ$=$PB$=a(1-\tan\theta)$; (iv) $\tan(45°+\theta)=\dfrac{1+\tan\theta}{1-\tan\theta}$.

CHAPTER V.

THREE DIMENSIONAL PROBLEMS.

Intersecting planes.

If we open a book at any angle, the two pages, if flat, form two planes intersecting in a straight line, the line of the binding. Draw a straight line on each page at right angles to the line of the binding and intersecting on that line. Then the angle between the two planes formed by the pages is defined as the angle between these two straight lines.

The following statements are important :

(i) *Any two planes intersect in a straight line unless they are parallel.*

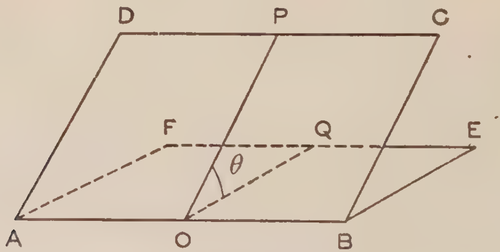


FIG. 125.

(ii) *If* AB *is the line of intersection of two planes* ABCD, ABEF *and from any point* O *on* AB *lines* OP, OQ *are drawn in the two planes perpendicular to* AB, *then the angle* POQ *is by definition equal to the* **angle between the two planes.**

(ii) *If the plane* ABEF *is horizontal, the line* OP *is called* a **line of greatest slope** *of the plane* ABCD.

Note: all lines of greatest slope in a plane are parallel; each represents the steepest and shortest path up hill.

Intersecting line and plane.

Suppose any line OP cuts a plane ABCD at O; draw PN perpendicular to the plane ABCD to cut it at N; join ON and

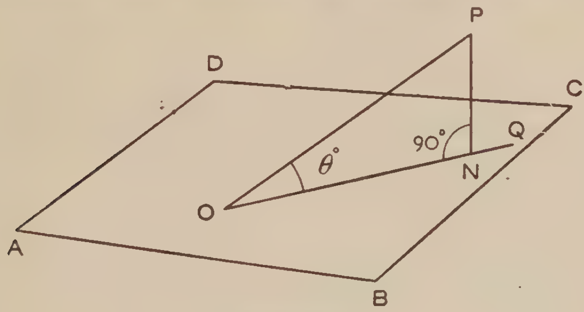


FIG. 126.

produce to Q. *Then the angle* POQ *is defined as the angle between the straight line* OP *and the plane* ABCD.

Note: (i) Since PN is perpendicular to the plane ABCD, it is perpendicular to every line in the plane; thus $\angle PNO = 90°$. For example, if ABCD is a horizontal plane, PN is a vertical line, and every line in ABCD is a horizontal line and so is perpendicular to PN.

(ii) if $\angle POQ = \theta$, $ON = OP \cos\theta$ and $NP = OP \sin\theta$.

(iii) ON is the *projection* of OP on the plane ABCD.

General procedure. Three-dimensional problems are usually solved by taking a succession of triangles in different planes and applying to each separately the results which have already been established.

In order to calculate the angle between two planes, it is usually necessary to take (or construct) two lines perpendicular to the

line of intersection of the planes and consider some triangle to which these two lines belong.

In order to calculate the angle between a line and a plane, it is usually necessary to take (or construct) the projection of the line on the plane and consider some triangle to which the line and its projection belong.

Part of the initial difficulty occurs in drawing suitable figures. A perspective figure should first be drawn with the dimensions clearly marked. The student may then draw separately the triangles used in the working, as in Example I. below, so as to see more clearly which angles in the perspective figure are right angles. But he should as soon as possible acquire the habit of working only with a perspective figure, as in Examples II. and III. below.

When possible the problem should be illustrated by a simple model; *e.g.* the cover of a book can be tilted to represent an inclined plane, a match box can be used to represent a room, etc.; skeleton solids made from thin rods are very instructive.

Example I. A hall is 16 ft. long, 12 ft. wide, 8 ft. high.

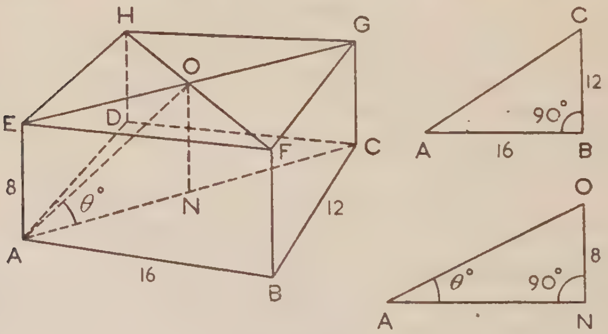


FIG. 127.

Calculate the angle which a cord stretched from the centre of the ceiling to one corner of the floor makes with the floor.

O is the centre of the ceiling and OA is the cord. N is the centre of the floor and ON is perpendicular to the floor.

$\therefore$ AN is the projection of AO.

Then $\angle$ OAN is the required angle.

By Pythagoras, $AC^2 = 16^2 + 12^2 = 256 + 144 = 400$;

$$\therefore AC = 20 \text{ ft.};$$

$$\therefore AN = 10 \text{ ft., also } ON = 8 \text{ ft.};$$

$$\therefore \tan OAN = \tfrac{8}{10} = 0{\cdot}8;$$

$$\therefore \angle OAN = 38° \ 40'.$$

Example II. With the data of the above Example, calculate the angle between the planes OAB, ABCD.

AB is the line of intersection of the two planes: we therefore look for two lines perpendicular to AB in the two given planes.

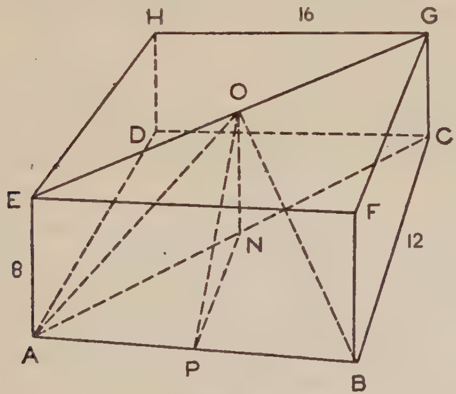


FIG. 128.

Let P be the mid-point of AB.

Then PO is perpendicular to AB, because OAB is an isosceles triangle; also PN is perpendicular to AB;

$\therefore$ $\angle$ OPN is the required angle.

Now $ON = 8$ ft., $PN = \frac{1}{2}BC = 6$ ft.; $\therefore$ $\tan OPN = \frac{8}{6} = 1{\cdot}3333$;

$$\therefore \angle OPN = 53° \ 8'.$$

EXERCISE V. a.

1. A man holds one end of a pole 8 ft. long in his hand and the other end rests on level ground. His hand is 3 ft. above the ground. What angle does the pole make with the ground ?

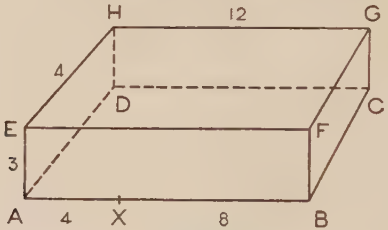


FIG. 129.

With the notation of Fig. 129, which represents a box with rectangular faces, find the angle of inclination in Nos. 2-12.

2. AG and plane ABCD.

3. HB and plane HDAE.

4. HB and plane DHGC.

5. HX and plane ABCD.

6. Planes ABCD and ABGH.

7. Planes EHCB and FBCG.

8. Planes HDX and HDAE.

9. Planes HEX and ABCD.

10. Planes HDX and HDB.

11. Lines BH and AG.

12. Lines HX and GX.

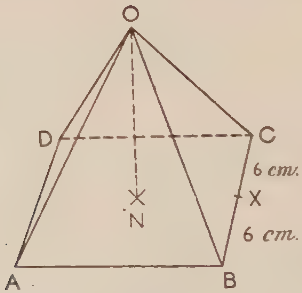


FIG. 130.

With the notation of Fig. 130 which represents a right pyramid of height 7 cm. on a square base ABCD, side 12 cm., find the angles of inclination in Nos. 13-20.

Note.—In a *right* pyramid the perpendicular ON from the vertex O to the base meets the base at its centre N.

13. OB and ABCD. **14.** OX and ON.
15. Planes OBC and ABCD. **16.** OA and OC.
17. OD and OC. **18.** Planes ONB and ONX.
19. Planes OBC and OAD. **20.** ON and plane OAB.

21. In Fig. 130, draw CE perpendicular to OB; calculate the length of CE; hence find the angle between the planes OBC and OBA.

22. The edges of a box are 5, 6, 7 inches; find the angle a diagonal of the box makes with the largest face.

23. Two vertical poles 6 ft., 10 ft. high stand on level ground 5 ft. apart on an East-West line; a tight rope connects their tops; when the sun is due South the shadow of the shorter pole is $4\frac{1}{2}$ ft. long. Find the bearing of the shadow cast by the rope.

24. A flagstaff AP stands at the corner A of a rectangular court ABCD; AB = 80 yd., BC = 60 yd.; the angle of elevation of P from B is 12° 30′. Find the angle of elevation of P from C and D.

25. A wall 12 ft. high runs east and west; the sun bears S. 60° W. at an elevation of 32°. Calculate the *breadth* of the shadow of the wall on the ground.

26. Find the *breadth* of the shadow in No. 25, if, further on, the wall runs north and south.

27. A ring, radius 2 ft., is suspended from a point by eight equal strings, each 3 ft. long, attached symmetrically to the ring. Find the angle between two consecutive strings.

28. ABC is an equilateral triangle inscribed in a circle, centre O, radius 80 ft., on a horizontal plane; a mast OE of length 60 ft. is fixed vertically at O, and stayed by wires from E to A, B, C. Calculate $\angle$AEB. Find also the angle between the planes BEC and BAC.

29. The elevation of the top of a tower is 45° from each of two points on the ground 200 ft. apart, one due South and the other due East of the tower. What is the height of the tower?

30. From Tirywen, due South of the Sugar Loaf Mountain, the elevation of the peak is 9° 26′; from Llangrwyne 2·2 miles due West of Tirywen and at the same level, 200 ft. above the sea, the angle of elevation of the peak is 6° 19′. Find the height of the peak above sea level.

31. The base of a pyramid is an equilateral triangle ABC of side 2 inches, and one of the faces is also an equilateral triangle OAB at right angles to the base. Find the sides and angles of the other two triangular faces OAC, OBC.

Find also the lengths of the perpendiculars from O to AB and AC, and hence find the angle between the planes BAC and OAC.

32. A man observes that the bearing of a chimney is N. 70° W.; after walking 100 yd. S.W. he finds that the bearing is N.W., and that the angle of elevation of its summit is 6° 20′. Find the height of the chimney.

33. The base of a pyramid is a regular hexagon of side 8 cm. and its height is 6 cm. Find (i) the inclination of each slant edge to the base, (ii) the angle between each face and the base, (iii) the angle between two adjacent faces.

34. The base of a right pyramid is a square of side 4 in.; each face makes an angle of 53° with the base. Find (i) the height of the pyramid, (ii) the angle each slant edge makes with the base.

35. The base of a right pyramid is a regular pentagon; each face is an equilateral triangle. Find the angle which (i) each face, (ii) each edge makes with the base.

Example III. A hill-side is a plane sloping at 27° to the horizontal; a straight track runs up the hill at an angle of 34° with a line of greatest slope. What angle does the track make with the horizontal?

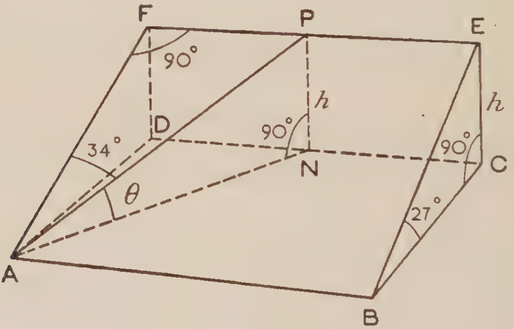


FIG. 131.

AB is the line of intersection of the hill-side and a horizontal plane ABCD; AF, BE are lines of greatest slope meeting a horizontal line at F, E.

Let the track AP cut EF at P; draw PN, EC perpendicular to the horizontal plane ABCD.

Then AN is the projection of AP on ABCD; it is required to find $\angle$ PAN $=\theta°$, say.

We have $\angle$ FAP $=34°$, $\angle$ AFP $=90°$, $\angle$ EBC $=27°$.

Let PN $=h$ ft., then EC $=h$ ft.

From the right-angled $\triangle$ ECB, BE $=h$ cosec 27° ft.;

$$\therefore \text{AF}=\text{BE}=h \operatorname{cosec} 27° \text{ ft.}$$

From the right-angled $\triangle$ AFP,

$$\text{AP}=\text{AF} \sec 34° = h \operatorname{cosec} 27° \sec 34°.$$

From the right-angled $\triangle$ ANP,

$$\sin\theta° = \frac{\text{PN}}{\text{AP}} = \frac{h}{h \operatorname{cosec} 27° \sec 34°} = \sin 27° \cos 34°;$$

$$\therefore \sin\theta° = 0{\cdot}4540 \times 0{\cdot}8290 \simeq 0{\cdot}3764;$$

$$\therefore \theta° = 22°\ 7'.$$

Compass bearings. The "bearing" of a *horizontal* line has already been defined (p. 3). A further definition is required for lines which are not horizontal.

Suppose in Fig. 131, where ABCD is a horizontal plane, that AD points due North; then AFD is the vertical plane which contains the line through A pointing North, and APN is the vertical plane containing AP. *The bearing of* AP *is defined as the angle between the vertical plane containing* AP *and the vertical plane containing the line through* A *pointing North, i.e.* the angle between the planes APN and AFD, and this is equal to $\angle$ NAD. It is important to notice that it is *not* equal to $\angle$ PAF, see Example IV. The reader will see that the bearing of the line AP is the angle which the *projection* of AP on a horizontal plane makes with a line in the plane pointing North.

Example IV. If, with the data of Example III. above, the lines of greatest slope of the plane ABEF bear due North, find the bearing of the track AP.

Let the required angle $\angle$ NAD $=\phi°$.

Then DN $=$ FP $=$ FA tan 34°, since $\angle$ PFA $=90°$.

But $FA = DA \sec 27°$, since $\angle FDA = 90°$, $\angle FAD = 27°$;

$$\therefore\ DN = DA \sec 27° \tan 34°;$$

$$\therefore\ \tan \phi° = \frac{DN}{DA} = \sec 27° \tan 34°, \text{ since } \angle NDA = 90°$$

$$= 1{\cdot}1223 \times 0{\cdot}6745 \simeq 0{\cdot}7570;$$

$$\therefore\ \phi° = 37°\ 7';$$

$\therefore$ the bearing of AP is **37° 7′**.

Note that this is *not* equal to $\angle$ PAF, which is 34°.

EXERCISE V. b.

1. A rectangular sheet of paper ABCD lies flat on the face of a desk which slopes at 20° to the horizontal; the lower edge AB is horizontal; AB = 8 in., BC = 6 in. (i) What is the slope of AC? (ii) What is the slope of a line on the paper making 72° with AB? (iii) A line AP is drawn on the paper with a slope of 15°; what is $\angle$ PAB?

2. XY is the axis and AB a generator of a circular cylinder, diameter 4 in., height 12 in.; YP is a radius of the base, such that $\angle BYP = 50°$. Calculate the angle which AP makes with the base.

FIG. 132.

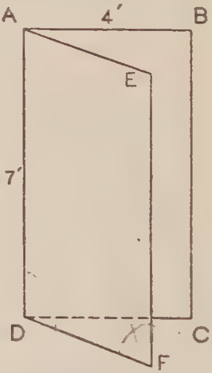


FIG. 133.

3. Fig. 133 represents a door opening through an angle of 38°; find the angle between AC and AF.

4. With the data of No. 3, find the angle which AD makes with the plane ACF.

5. BC, the base of the isosceles triangle ABC, is horizontal; $\triangle$ABC makes an angle of 70° with the horizontal; $\angle$BAC = 34°. Find the angle of slope of AB.

6. A book is 4 in. wide, 7 in. high; it is placed flat on a level table and the cover is opened so as to make an angle of 125° with the first page. What is the angle of slope of a diagonal of the cover?

7. With the data of No. 6, find the angle through which the cover is opened if the angle of slope of a diagonal of the cover is 25°.

8. A blackboard on an easel slopes at 72° with the ground; a line is drawn on the board making 38° with a horizontal edge of the board. What angle does this line make with the ground?

9. A watch lies on a stand which makes 70° with the horizontal: the hour hand is horizontal at 3 o'clock. What is its angle of slope at (i) one o'clock, (ii) eight o'clock?

10. A man zig-zags in ascending a road of gradient 1 in 10 [*i.e.* $\sin^{-1}(\frac{1}{10})$]; his path makes an angle of 40° with a line of greatest slope. What is the gradient of the path he follows?

11. In $\triangle$ABC, AB = 5, BC = 4, CA = 3; the triangle is rotated about AB through 35°. Find the angle between the old and new positions of AC.

12. With the data of No. 11, find the angle through which the triangle is rotated about AB if the new and old positions of BC are inclined to each other at 50°.

13. A rectangular sheet of paper ABCD is folded about BD so that the new position BED of BCD is perpendicular to its old position; AB = 6 in., BC = 8 in. Find the angle which AE makes with the plane ABD.

14. A hill, facing due North, slopes at an angle of 18° with the horizontal, and a road is made on its face bearing N. 57° E. Find the angle of slope of the road.

15. With the data of No. 14, find the bearing of a path up the hill if the gradient of the path is 1 in 5 [*i.e.* $\sin^{-1}(\frac{1}{5})$].

16. The bearing of a line of greatest slope of a hill is N. 72° E., and its angle of slope is 21°; a track running South of East makes 39° with a line of greatest slope. What is its bearing?

17. Four equal panes of glass in the shape of trapeziums, with parallel sides 14 in., 4 in. long and slant sides each 10 in. long, are joined together to form the cover of a street gas-lamp. What is the angle between (i) each pane and the vertical, (ii) two adjacent panes?

18. A hill slopes up at an angle of 28° with the horizontal. A skier with skins can climb at an angle of 12° with the horizontal, and one without skins only at an angle of 5° with the horizontal. What is the angle between their tracks on the hill ?

19. The roofs of the buildings along two adjacent sides of a rectangular court make angles 30°, 45° with the horizontal. What is the angle of slope of the gutter running down the line of intersection of the roofs ?

20. ABC is an isosceles triangle in a vertical plane with its base BC horizontal; its shadow on the horizontal plane is the triangle PBC; if PB = PC and $\angle$ BPC = $2\beta°$ and $\angle$ BAC = $2\alpha°$, prove that the sun's elevation is $\tan^{-1}(\cot \alpha . \tan \beta)$.

21. A rod AB, 6 inches long, is suspended by two equal vertical strings EA, FB, each 10 in. long from fixed points E, F at the same level. The rod is now twisted so that its mid-point O rises vertically

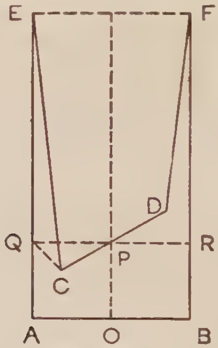


FIG. 134.

one inch, the rod remaining horizontal. What is the angle of twist ? [CPD is the new position of AOB ; if QPR is drawn parallel to AB, $\angle$EQC = 90° ; required to find $\angle$QPC.]

CHAPTER VI.

GRAPHICAL METHODS.

Limit ratios. Draw a circle, centre O, of *unit* radius OA; draw a radius OP and produce it to meet the tangent at A in T; draw PN perpendicular to OA.

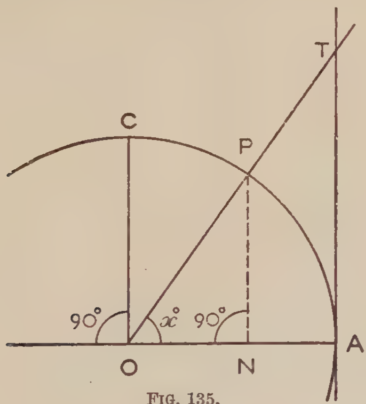


FIG. 135.

Suppose $\angle AOP = x°$.

Then we have seen that

$$\sin x° = \frac{NP}{OP} = \frac{NP}{1},$$

(*i.e.* the number of units of length in NP),

$$\cos x° = \frac{ON}{OP} = \frac{ON}{1},$$

$$\tan x° = \frac{AT}{OA} = \frac{AT}{1}.$$

The Angle 0°.

Draw a figure similar to Fig. 135, making x very small. [*The reader should draw such a figure and consider what happens to the lengths of* NP, ON, AT.]

We see that the nearer x approaches the value 0, the smaller NP and AT become, while ON approaches the value 1.

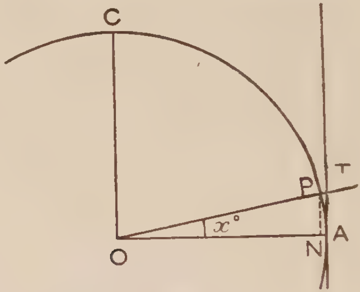


FIG. 136

And we say that, in the limit,

$$\sin 0° = 0, \quad \cos 0° = 1; \quad \tan 0° = 0.$$

Note. (i) The formula $\sin^2 x + \cos^2 x = 1$

shows that if $\cos x = 1$, then $\sin x$ must equal 0.

(ii) The formula $\tan x = \dfrac{\sin x}{\cos x}$ shows that

$$\tan 0° = \frac{\sin 0°}{\cos 0°} = \frac{0}{1} = 0.$$

The Angle 90°.

Now draw a figure similar to Fig. 135, making x very nearly 90. [*The reader should draw such a figure and make his own deductions as before.*]

We see that the nearer x approaches the value 90, the smaller ON becomes, while NP approximates to OC and AT increases indefinitely. When $x = 90$, N coincides with O, P coincides

with C, and OP has become parallel to AT. We therefore say that, in the limit,

$$\sin 90° = 1, \quad \cos 90° = 0, \quad \tan 90° \text{ is } \infty.$$

Note. (i) The symbol ∞ is used for the word "infinity"; the statement $\tan 90°$ is ∞ is conventional, *i.e.* it does not

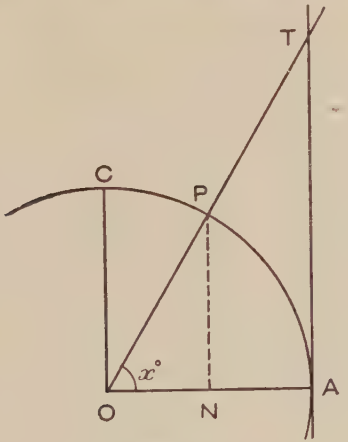


FIG. 137.

imply any numerical equality; it merely means that the tangent of an acute angle can be made to exceed any named value by taking x sufficiently near 90°. Thus

$$\tan 84° 18' > 10; \quad \tan 87° 12' > 20; \quad \tan 88° 54' > 50;$$
$$\tan 89° 30' > 100; \quad \tan 89° 54' > 500; \text{ etc.}$$

(ii) It is suggested that some **oral** work should be based on the application of the formulae

$$\sin^2 x + \cos^2 x = 1; \quad \tan x = \frac{\sin x}{\cos x}; \quad \sin x = \cos(90° - x);$$
$$\cos x = \sin(90° - x); \quad \tan(90° - x) = \cot x = \frac{1}{\tan x}$$

to the six results obtained above.

(iii) The values of cosec x°, sec x°, cot x°, when $x=0$ or 90 should also be discussed orally, using the definitions $\text{cosec}\, x = \frac{1}{\sin x}$, etc.

EXERCISE VI. a. (Oral.)

Obtain from the Tables the values of the ratios in examples Nos. 1-12.

1. sin 89°. **2.** cosec 89°. **3.** cos 89°. **4.** sec 89°.
5. tan 89°. **6.** cot 89. **7.** sin 30′. **8.** cosec 30′.
9. cos 30′. **10.** sec 30′. **11.** tan 30′. **12.** cot 30′.

What can you say about an acute angle x° in the following examples, Nos. 13-24.

13. $\sin x^\circ > 0{\cdot}9999$. **14.** $\cos x^\circ < 0{\cdot}01$. **15.** $\tan x^\circ > 25$.
16. $\cos x^\circ > 0{\cdot}9999$. **17.** $\sin x^\circ < 0{\cdot}01$. **18.** $\cot x^\circ > 25$.
19. $\text{cosec}\, x^\circ > 100$. **20.** $\sec x^\circ < 1{\cdot}0001$. **21.** $\cot x^\circ < 0{\cdot}01$.
22. $\text{cosec}\, x^\circ < 1{\cdot}0001$. **23.** $\sec x^\circ > 100$. **24.** $\tan x^\circ < 0{\cdot}01$.

25. Make a Table similar to the given Table showing values of cosec x°, sec x°, cot x°.

x	$\sin x^\circ$	$\cos x^\circ$	$\tan x^\circ$
0°	0	1	0
90°	1	0	∞

26. Using the table in No. 25, what do you deduce from

$$\cot x^\circ = \tan (90^\circ - x)$$

if (i) $x=0$, (ii) $x=90$?

Graphs of sin x° and cos x°.

The variation in value of $\sin x^\circ$ and $\cos x^\circ$, as x varies from 0 to 90, can be illustrated by drawing their graphs. The Tables may be used to give the necessary values :

x	0	10	20	30	40	50	60	70	80	90
$\sin x^\circ$ (to 2 figures)	0	0·17	0·34	0·50	0·64	0·77	0·87	0·94	0·98	1
$\cos x^\circ$ (to 2 figures)	1	0·98	0·94	0·87	0·77	0·64	0·50	0·34	0·17	0

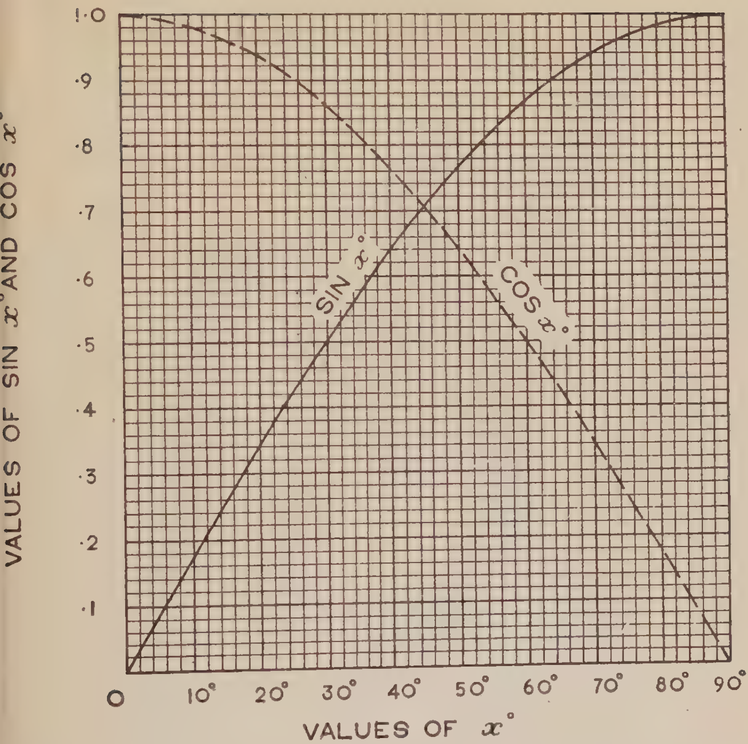


FIG. 138.

Note. When plotting a graph of any function of x, *always* draw the x-axis from left to right *across* the page.

EXERCISE VI. b. (Oral.)

1. If you hold Fig. 138 in front of a looking-glass what can you say about the image of the graph of (i) sin x°, (ii) cos x° ?

2. Use Fig. 138 to read off the values of

(i) sin 18° ;	(ii) sin 36° ;	(iii) sin 72° ;
(iv) cos 18° ;	(v) cos 36° ;	(vi) cos 72°.

3. Use Fig. 138 to solve the following equations:
(i) $\sin x° = 0{\cdot}56$; (ii) $\cos x° = 0{\cdot}24$; (iii) $\sin x° = 0{\cdot}83$;
(iv) $\cos x° = 0{\cdot}90$; (v) $\sin x° = \cos x°$.

4. Can you use the graph of $\sin x°$ to solve the equation
$$\cos x° = 0{\cdot}60\ ?$$

5. Can you use the grapn of $\cos x°$ to solve the equation
$$\sin x° = 0{\cdot}74\ ?$$

Graphs of tan x° and cot x°.

The variation in value of $\tan x°$ and $\cot x°$ as x increases from 0 may also be illustrated graphically, but the values

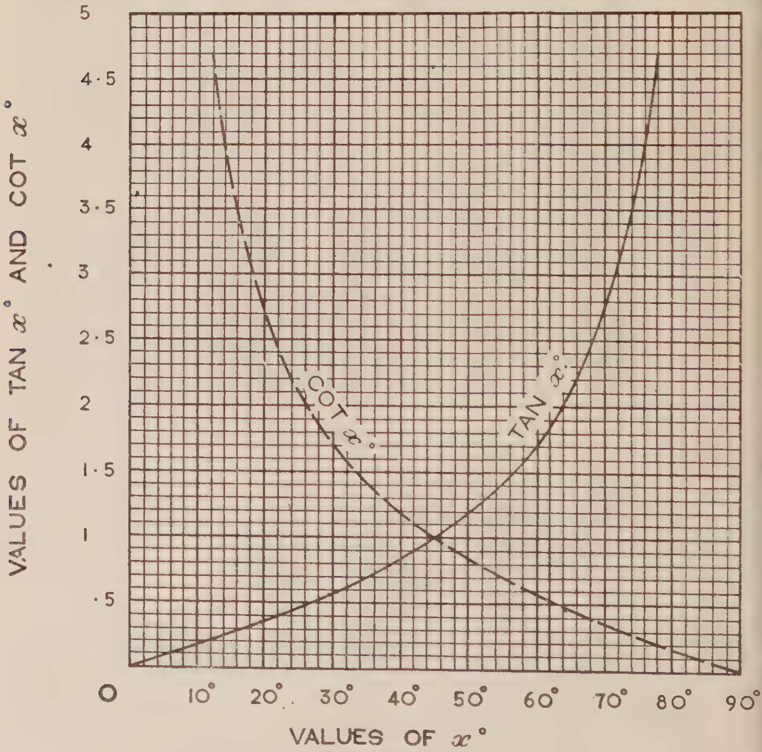


Fig. 139

of tan $x°$, when x approaches 80 from below, and the values of cot $x°$ when x approaches 10 from above, cannot be shown on the graph without making the scale very small. As before, the Tables may be used to give the necessary values.

x	0	12	15	20	30	40	50	60	70	75	78	90
tan $x°$	0	0·21	0·27	0·36	0·58	0·84	1·19	1·73	2·75	3·73	4·70	
cot $x°$		4·70	3·73	2·75	1·73	1·19	0·84	0·58	0·36	0·27	0·21	0

EXERCISE VI. c. (Oral.)

1. Suppose the graph of tan $x°$ is drawn on transparent paper. If you hold it up to the light with the back of the paper towards you what can you say about the view you get of the graph ?

2. What is the reflection of the graph of cot $x°$ in a looking-glass if the paper is parallel to the face of the glass ?

3. Why are there two blank spaces in the table of values for tan $x°$ and cot $x°$ above ?

4. Use Fig. 139 to read off the values of :
(i) tan 24° ; (ii) tan 56° ; (iii) tan 74° ;
(iv) cot 16° ; (v) cot 55° ; (vi) cot 76°.

5. Use Fig. 139 to solve the following equations :
(i) tan $x° = 0·23$; (ii) cot $x° = 0·51$; (iii) tan $x° = 3$;
(iv) cot $x° = 4$; (v) tan $x° =$ cot $x°$.

6. Can you use the graph of tan $x°$ to solve the equation

$$\cot x° = 0·70\ ?$$

7. Can you use the graph of cot $x°$ to solve the equation

$$\tan x° = 1·1\ ?$$

Graphical applications.

As in Algebra, the applications of graphical methods which arise most frequently are (i) the solutions of equations, (ii) the determination of maxima and minima values of given functions. These are best illustrated by an example.

Example. Draw the graph of $3 \sin x^\circ + 2 \cos x^\circ$ for values of x from 30 to 80, and use it to find (i) a maximum value of $3 \sin x^\circ + 2 \cos x^\circ$, (ii) any solutions of the equation

$$3 \sin x^\circ + 2 \cos x^\circ = 3{\cdot}4$$

in that range.

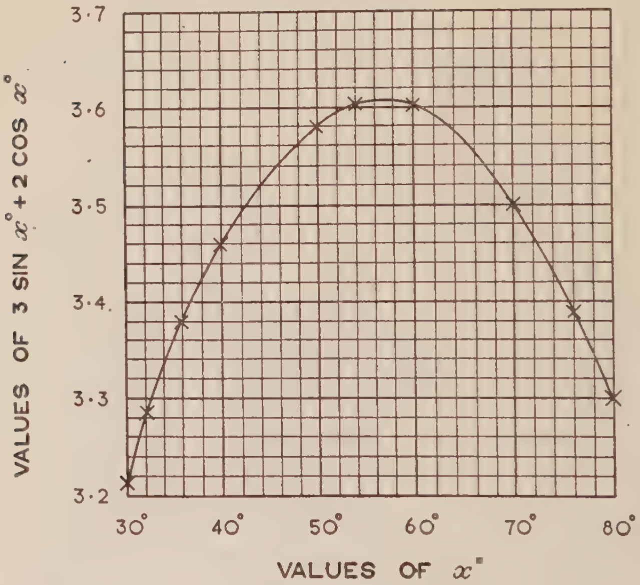


FIG. 140.

Using the Tables, we obtain the following values

x	30	40	50	60	70	80	32	36	54	76
$3 \sin x^\circ$	1·5	1·929	2·298	2·598	2·820	2·955	1·590	1·764	2·427	2·911
$2 \cos x^\circ$	1·732	1·532	1·286	1·0	0·684	0·348	1·696	1·618	1·176	0·484
$3 \sin x^\circ + 2 \cos x^\circ$	3·232	3·461	3·584	3·598	3·504	3·303	3·286	3·382	3·603	3·395

(i) From the graph we see that the maximum value of the function is approximately 3·605, corresponding to $x \simeq 56$.

(ii) The graph crosses the 3·4 level when $x \simeq 37$, and when $x \simeq 75$.

∴ 37° and 75° are approximate solutions of the equation

$$3 \sin x° + 2 \cos x° = 3·4.$$

Note. When making the Table of Values it is natural to begin by taking $x = 30, 40, 50, 60, 70, 80$. But when these are plotted they are seen to be insufficient for drawing the figure accurately. We therefore take the extra values 32, 36, 54, 76 and add them at the end of the Table.

EXERCISE VI. d.

1. Use Fig. 140 to read off the values of $3 \sin x° + 2 \cos x°$, when

(i) $x = 34$; (ii) $x = 45$; (iii) $x = 67$; (iv) $x = 78$.

2. Use Fig. 140 to obtain solutions of the equations:

(i) $3 \sin x° + 2 \cos x° = 3·5$; (ii) $3 \sin x° + 2 \cos x° = 3\frac{1}{3}$;
(iii) $6 \sin x° + 4 \cos x° = 6·9$; (iv) $3 \sin x° + 2 \cos x° = 3·25$.

3. Draw in one figure (as on p. 82), the graphs of

(1) cosec $x°$ for values of x from 10 to 90,
(2) sec $x°$ for values of x from 0 to 80.

(*a*) Read off the values of

(i) cosec 26°; (ii) cosec 44°; (iii) cosec 65°;
(iv) sec 25°; (v) sec 36°; (vi) sec 75°.

(*b*) Use your graphs to solve the equations

(i) cosec $x° = 3·9$; (ii) sec $x° = 3·4$; (iii) cosec $x° = 1·1$;
(iv) sec $x° = 1·2$; (v) sec $x°$ = cosec $x°$.

4. Draw the graph of $\sin(2x°)$ for values of x from 0 to 45. Compare the result with Fig. 138, p. 81.

5. Draw the graph of $\cos(x° + 30°)$ for values of x from -30 to 60. Compare the result with Fig. 138, p. 81.

6. Draw the graph of $3 \sin x° + 4 \cos x°$ for values of x from 0 to 90. (i) Find a maximum value of the function; (ii) find solu-

tions of the equation $3 \sin x^\circ + 4 \cos x^\circ = 3{\cdot}5$; (iii) *sketch* the graph of $3 \cos x^\circ + 4 \sin x^\circ$ on the same figure; (iv) draw on the same figure the graph of $5 \sin (x^\circ + 53^\circ\, 8')$, and compare the result with the first graph.

7. Draw the graph of $\sin x^\circ - \cos x^\circ$ for values of x from 0 to 90. What is the acute angle whose sine exceeds its cosine by 0·3 ?

8. Draw the graph of $\tan 3x^\circ + \cot 2x^\circ$ for values of x from 10 to 20. Find a solution of the equation $\tan 3x + \cot 2x = 2{\cdot}9$.

9. Find the maximum value of $2 \cos \phi^\circ - \sin \theta^\circ$ if $\theta^\circ + \phi^\circ = 60^\circ$, and the value of θ for which the expression is a maximum.

10. Draw with the same scale and axes the graphs of $\operatorname{cosec}(x^\circ + 15^\circ)$ and $\sec x^\circ$ for values of x from 0 to 60. Find a solution of the equation $\operatorname{cosec}(x^\circ + 15^\circ) = \sec x^\circ$. Prove that the correct answer is given by $x + 15 = 90 - x$.

11. Draw the graph of $\sin x^\circ + \sin (2x^\circ)$ for values of x from 0 to 45; hence find a solution of the equation $\sin x^\circ + \sin (2x^\circ) = 1$.

12. Draw the graph of $x \sin x^\circ$ for values of x from 0 to 90; hence find a solution of the equation $x \sin x^\circ = 40$.

13. Tabulate the values of $\sin x^\circ$ when x has the values 31, 31·1, 31·2, ... 31·9, 32. Plot the results and read off from the graph the difference between (i) $\sin 31^\circ\, 6'$ and $\sin 31^\circ\, 9'$; (ii) $\sin 31^\circ\, 48'$ and $\sin 31^\circ\, 51'$.

(*a*) What do the difference columns in the Tables give for a difference of 3′ ?

(*b*) What is the general effect if a small portion of Fig. 138 in the neighbourhood of $x = 31$ is examined with a magnifying glass ?

14. Tabulate the values of $\tan x^\circ$, (i) when x has the values 31, 31·1, 31·2, ... 31·9, 32; (ii) when x has the values 87, 87·1, 87·2, ... 87·9, 88. Plot the results in two separate graphs, choosing for each the most suitable scale.

(*a*) What is the chief difference between the graphs ?

(*b*) Read off the values of

(i) $\tan 31^\circ\, 9' - \tan 31^\circ\, 6'$ and $\tan 31^\circ\, 51' - \tan 31^\circ\, 48'$;

(ii) $\tan 87^\circ\, 9' - \tan 87^\circ\, 6'$ and $\tan 87^\circ\, 51' - \tan 87^\circ\, 48'$.

(*c*) What do the difference columns in the Tables give for a difference of 3′, and why ?

(*d*) What is the general effect if a small portion of Fig. 139 in the neighbourhood of (i) $x = 31$; (ii) $x = 80$ is examined with a magnifying glass ?

Find graphically a solution of the following equations:

15. $\sin 2x° = \cos 3x°$. **16.** $2 \sin x° = \sin 3x°$.

17. $3 \sin x° = \sin (x° + 36°)$. **18.** $30 \tan x° = x$.

19. $\tan x° + 2 \cot x° = 3\frac{3}{4}$. **20.** $1 + \sec (2x°) = \frac{x}{10}$.

21. Fig. 141 represents two corridors 9 ft., 6 ft. wide meeting at right angles. Show that AB equals $9 \operatorname{cosec} \theta + 6 \sec \theta$ feet. Hence

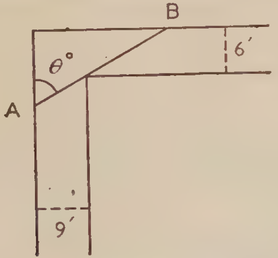


FIG. 141.

find graphically the length of the longest pole that can be carried round the corner without tilting it.

22. Find graphically the value of x for which $\frac{60}{x} + \tan x°$ is a minimum.

23. A uniform rod AB, 8 inches long, rests on a peg P with the end B against a smooth vertical wall; P is 1 inch from the wall. If the rod makes an angle $\theta°$ with the horizontal, show that the height of

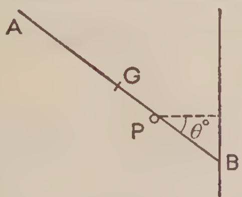


FIG. 142.

its mid-point G above the level of P is $(4 \sin \theta° - \tan \theta°)$ inches. Find graphically the value of θ for which this is greatest. [This corresponds to the position of equilibrium.]

24. A string 6 ft. long rests over two pegs A, B at the same level 2 ft. apart; a body of weight 4 lb. is attached to its mid-point and bodies each of weight 3 lb. are attached to its ends. [See Fig. 143.]

In the position shown in Fig. 143 the depth of the centre of gravity of the system below AB is $\frac{1}{5}(2 \tan \theta° - 3 \sec \theta° + 9)$ feet. Find

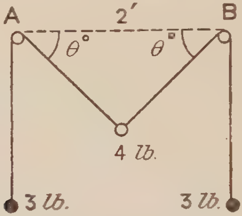


FIG. 143.

graphically the value of θ for which this depth is a maximum. [This corresponds to the position of equilibrium.]

REVISION PAPERS. R. 7-13.

R. 7.

1. A tower 300 ft. away subtends an angle of 25° at a point 25 ft. above the foot of the tower. Calculate the height of the tower.

2. ABCD is a rectangle. Calculate $\angle$APB.

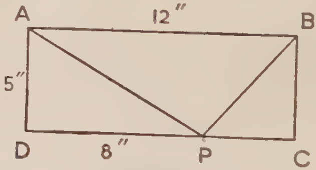


FIG. 144.

3. Evaluate (i) $\sin^2 45°$; (ii) $\tan 60° \cot 30°$.

4. Find a value of x such that (i) $\cos 4x° = \sin 5x°$;
(ii) $\sec (x+10)° = \operatorname{cosec} (2x-10)°$.

5. In $\triangle$ABC, AB$=3$, BC$=4$, CA$=5$; the triangle is rotated about AB through 60°. Find the angle between the old and new positions of AC.

R. 8.

1. If $\tan x° = \frac{1}{2}$, calculate $\sec x°$ without using Trigonometric Tables. Compare your result with that obtained direct from the Tables.

2. Find the length of the longer diagonal of a rhombus whose side is 5 in. and acute angle 41°.

3. Fig. 145 represents a rectangular table touching two walls of a room. Find the distance of D from OB and of C from OA.

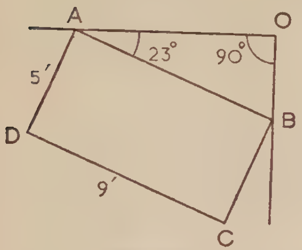


FIG. 145.

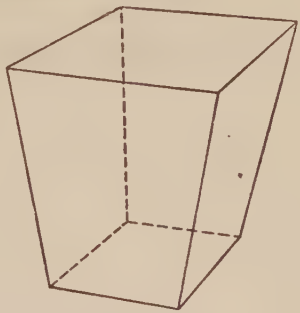


FIG. 146.

4. Find θ if (i) $\sin\theta° = 2\sin 30°$; (ii) $\operatorname{cosec}\theta° = 2\operatorname{cosec} 30°$.

5. The cover of a gas-lamp is in the shape of a frustum of a pyramid on a square base; the bottom is a square of side 4 in. and the top a square of side 8 in.; the top and bottom are 10 in. apart. Find the inclination of the edges to the vertical. (Fig. 146.)

R. 9.

1. A man walks 3 miles N.E., then 5 miles N., then 2 miles N. 25° E. Find how far he then is (i) East, (ii) North of his starting point.

2. Evaluate (i) $\operatorname{cosec} 45° \sec 45°$; (ii) $\dfrac{\sec 30° \operatorname{cosec} 60°}{\sin 30°}$.

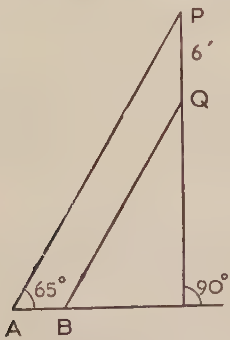


FIG. 147.

3. In Fig. 147 AP is parallel to BQ. Find AB.

4. Two circles, radii 6 cm. and 10 cm., have their centres 20 cm. apart. Find the angle made with the line of centres, (i) by their direct common tangents, (ii) by their transverse common tangents.

5. A pyramid 6 inches high has a square base, side 4″; its faces are equal. Find the inclination of each face to the base.

R. 10.

1. In Fig. 148 find AN and NC.

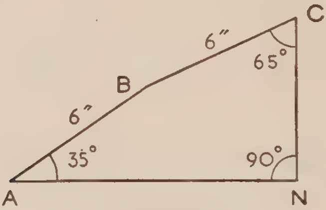


FIG. 148.

2. What can you say about an acute angle $x°$, if

(i) $\tan x° > 2$; (ii) $\sec x° > 2$; (iii) $\sin x° > \cos x°$?

3. A man standing on a cliff 120 ft. high observed two boats in the same vertical plane as himself. The angles of depression are 18°, 35°. How far apart are the boats?

4. Draw with the same scale and axes the graphs of $\sin x°$ and $\tan x°$ for values of x from 0 to 15. What can you deduce from these graphs?

5. OA, OB are edges of the rectangular floor of a room and OC is vertical. P is a point on the floor 2 ft. from OA and 3 ft. from OB. Q is a point on OC 4 ft. above O. What angle does the line PQ make with the floor?

R. 11.

1. If $\cos\theta = \frac{5}{7}$, calculate $\sin\theta$ without using Trigonometric Tables. Compare your result with that obtained direct from the Tables.

2. The sun is due South at elevation 47°; a telegraph pole 20 ft. high is 12 ft. away from a vertical wall running East and West, and on the south side of it. What is the length of the shadow of the pole on the wall?

3. With the data and figure of Ex. III. b., No. 6, p. 46, if

$$NB = 5 \text{ cm.}, \quad \theta° = 27°,$$

find the radius of the circle.

4. The crank CP rotates about C, and the end A of the connecting rod AP moves along the line CB. Find θ, ϕ, (i) when CA is $2a''$; (ii) when $\angle CPA = 90°$.

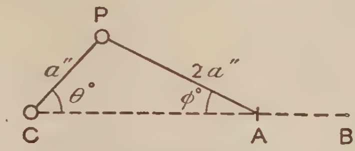


FIG. 149.

5. A cube, edge 10 cm., rests with its face ABCD on a horizontal table; it is tilted about AB until the face ABCD is inclined at 25° to the horizontal; AG is a diagonal of the cube. Find (i) the height of C and G above the table, (ii) the inclinations of AC and AG to the horizontal.

R. 12.

1. The deflection $\theta°$ of the needle of a galvanometer when a current of C amperes passes through the coil is given by $C = 0{\cdot}2 \tan \theta°$. Find the increase of current when the deflection increases from 15° to 35°.

2. Fig. 150 represents four rods, each of length 10 in., jointed together and suspended from points A, E on the same level. Find AE and the depth of C below AE.

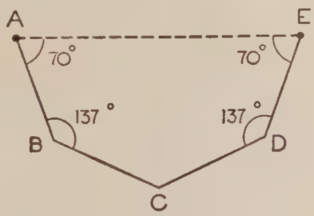


FIG. 150.

3. In Fig. 151 calculate $\angle PCQ$.

4. Draw the graph of $\sin x° + 3 \cos x°$ for values of x from 10 to 40, and find from it the maximum value of the expression.

5. ABCD is a rectangular court-yard surrounded by buildings 60 ft. high. When the sun is due South the shadow is represented

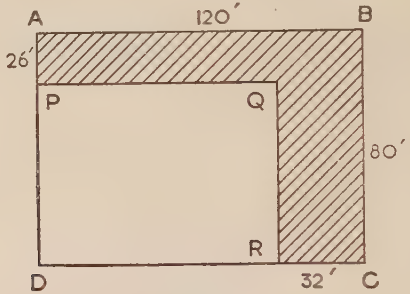


FIG. 151.

by the shaded portion of the figure. Find (i) the elevation of the sun, (ii) the bearing of B from A.

R. 13.

1. Find values for r and θ, given that

$$r \sin \theta^\circ = 7 \quad \text{and} \quad r \cos \theta^\circ = 11.$$

2. A case is being raised vertically through a hatchway CB as shown; the trap-door, CE = CB = 5 ft., rests against it. To what angle is CE opened ?

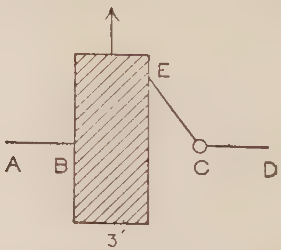


FIG. 152.

3. A regular pentagon is inscribed in a circle of radius 10 cm. Calculate the height of the minor segment cut off by one side.

4. With the notation and figure of Ex. IV. d., No. 13, p. 64. calculate c, given that $a = 10{\cdot}75''$, $d = 10''$, $b = 30''$, $\theta^\circ = 10^\circ$.

5. The rim of the bowl of an electric light is a circle of radius 8 in. It is suspended by three chains attached to points on the rim at the corners of an equilateral triangle, and to a hook in the ceiling 2 ft. above the plane of the rim. Find the inclination of each chain to the vertical.

R. 14.

1. Evaluate as shortly as possible

$$\frac{1}{\sin 17^\circ 35'} - \frac{1}{\cos 71^\circ 20'}; \quad \text{(ii) } \cos^2 63^\circ + \cos^2 27^\circ.$$

2. The tangents from a point A to a circle are each 7 cm. long and contain an angle of 152°. Find the distance of A from the centre of the circle.

3. A rod AB pivoted at A rests with B on a horizontal plane CD as shown. Find the height of B above the plane if AB is rotated about A through 100° in a vertical plane.

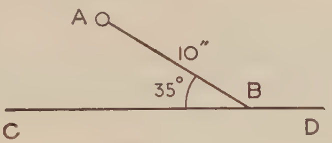


FIG. 153.

4. Draw the graph of $\tan x^\circ + 2\cot x^\circ$ for values of x from 35° to 70°. Find from the graph (i) the value of x for which $\tan x^\circ + 2\cot x^\circ$ is a minimum; (ii) two solutions of $\tan x^\circ + 2\cot x^\circ = 3{\cdot}2$.

5. A book lies on a level table; its cover 7″ by 5″ is open at an angle of 20° to the horizontal. What angle does a diagonal of the cover make with the horizontal if the cover turns about one of its longer sides?

R. 15.

1. Given $\sec\theta = b$, express $\sin\theta$ and $\tan\theta$ in terms of b.

2. A ball falls vertically from P and strikes a projecting ledge AB at Q and rebounds in the direction QR. If $\tan\phi^\circ = \frac{3}{4}\tan\theta^\circ$ and $\theta^\circ = 32°$, find the angle QR makes with the horizontal.

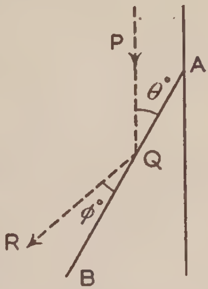


FIG. 154.

3. With the notation and figure of Ex. IV. d., No. 4, p. 62, if AB = 6 ft. and ∠ACB = 100°, find the distance the end P must be pulled down to increase ∠ACB by 40°.

4. A rectangular box is tilted as shown, so that the base makes an angle $\theta°$ with the horizontal AE. Show that the height of C above AE is $(p \sin \theta + q \cos \theta)$ inches.

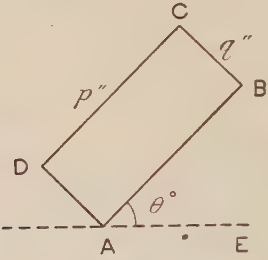


FIG. 155.

Hence determine the maximum value of $p \sin \theta + q \cos \theta$ for values of θ between 0 and 90.

If $p=5$, $q=2$, find the value of θ which gives the maximum value.

5. The greatest slope of a hill is 20° to the horizontal. What will be the slope of a path on the hill which makes an angle of 42° with a line of greatest slope?

R. 16.

1. If $\sin \theta = \dfrac{m^2-1}{m^2+1}$, express $\tan \theta$ and $\cos \theta$ in terms of m.

2. A ship A steams at 20 knots on a bearing 330°, and a ship B at 18 knots on a bearing of 250°. Find the distance of A, (i) North, (ii) East of B, 3 hours after they parted company. Find also the bearing of A from B at this time.

3. Fig. 156 represents a lamina which when suspended from A hangs so that G is vertically below A. Find the angle which BC makes with the horizontal.

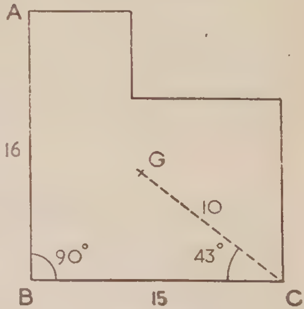


FIG. 156.

4. With the data of Ex. IV. d., No. 7, p. 63, find the greatest angle to which the gate can be opened if the weight jams at the top when it has risen $4\frac{1}{2}$ feet.

5. A rod 3 ft. long is suspended from the ceiling in a horizontal position by two equal vertical strings 5 ft. long attached to its ends.

The rod is then rotated through 90°, remaining horizontal, so that its centre C rises in a vertical line. Find the height C rises, and the angle which each string makes with the vertical in the new position.

R. 17.

1. Write down a value of θ, if

(i) $\dfrac{\sin \theta°}{\cos \theta°}=\dfrac{1}{\sqrt{3}}$; (ii) $\operatorname{cosec} \theta°=2$; (iii) $\tan \theta°=3 \cot \theta°$.

2. A piece of wire 3 ft. long is bent into the form of an isosceles triangle, of which one angle is 100°. Find the longest side.

3. A car is made for a cliff railway with wheels of diameters 60 in. and 20 in.; the centres of the wheels are 10 ft. apart, and the line joining them is horizontal when the car is on the rails. Find the inclination of the rails to the horizontal.

4. Draw the graph of $\operatorname{cosec} 2x°+\sec 3x°$ for values of x from 10 to 25, and find a solution of $\operatorname{cosec} 2x°+\sec 3x°=3{\cdot}5$.

5. The funnel of a steamer makes an angle of 80° with the deck. The steamer rolls, without pitching, through 10° on either side of the vertical. Find the extreme inclination of the funnel to the vertical.

R. 18.

1. If $x \cos \theta+y \sin \theta=4$ and $x \sin \theta-y \cos \theta=3$, find by squaring and adding a relation between x and y, independent of θ.

2. In Fig. 157, ABCD is a square of side 3 inches; find the distance of C from AP in two different ways. Hence prove that

$$\sin 20°+\cos 20°=\sqrt{2} \, . \sin 65°.$$

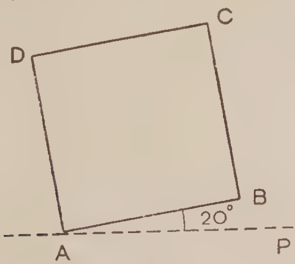


FIG. 157.

3. A flexible wire in the form of a square is bent into the form of a regular octagon. Find the percentage increase in the area enclosed.

4. From an upper window in a house which is 100 ft. from a church tower the angle of elevation of the top of the tower is 41°, and the angle of depression of the bottom is 15°. How high is the tower ?

5. A billiard-ball, diameter 5 cm., moves on a horizontal table along a line making 18° with a cushion, which overhangs so that the point at which the ball strikes it is 4 cm. above the table. Find the distance along the cushion between the point of contact and the point at the same height apparently aimed at.

CHAPTER VII.

ANGLES GREATER THAN A RIGHT ANGLE.

Coordinates. Fig. 158 represents two rectangular axes Ox, Oy drawn on inch-paper, *i.e.* unit of length one inch. The position of any point in the plane is fixed by its coordinates.

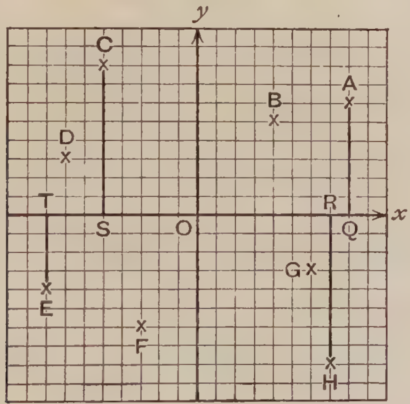


FIG. 158.

Thus A is $(+0{\cdot}8, +0{\cdot}6)$; C is $(-0{\cdot}5, +0{\cdot}8)$;
E is $(-0{\cdot}8, -0{\cdot}4)$; H is $(+0{\cdot}7, -0{\cdot}8)$.

The axes divide the plane into four quadrants; the object of attaching positive and negative signs to the coordinates is solely to distinguish between the quadrants; it is merely a convention, but a very useful one. It is therefore

necessary to distinguish between the coordinates of a point and the number of units of length in the distances of the point from the axes Ox, Oy. Thus, although the coordinates of C are $(-0{\cdot}5, +0{\cdot}8)$, the number of units of length in OS is 0·5, for OS = 0·5 in., and although the coordinates of E are $(-0{\cdot}8, -0{\cdot}4)$, the number of units of length in OT is 0·8, for OT = 0·8 in. and the number of units of length in TE is 0·4, for TE = 0·4 in.

EXERCISE VII. a. (Oral.)

Suppose the perpendiculars from B, D, F, G to Ox are BB′, DD′, FF′, GG′. See Fig. 158.

1. What are the coordinates of B and D? What are the lengths of the lines OB′, B′B, OD′, D′D?

2. Repeat No. 1 for the points F and G.

3. A point is 0·3 inch from Ox and 0·4 inch from Oy. Does this fix its position? What can you say about its position?

4. Mark the point $(+0{\cdot}3, -0{\cdot}4)$ on Fig. 158. Name, by a letter, some other point shown in the same quadrant.

5. Repeat No. 4 for the points $(-0{\cdot}3, +0{\cdot}4)$ and $(-0{\cdot}3, -0{\cdot}4)$.

6. The coordinates of a point K are (x, y); what do you know about x and y if K lies in the same quadrant as (i) C, (ii) H, (iii) A, (iv) E?

Note. The quadrants in which A, C, E, H *lie are called the first, second, third and fourth quadrants respectively.*

Trigonometrical ratios.

Take two rectangular axes Ox, Oy and imagine a line OP of fixed length r inches to rotate anti-clockwise about O, starting from the position Ox.

Suppose that at any time $\angle x\mathrm{OP} = \theta°$.

Draw PN perpendicular to Ox; let ON $= a$ in., NP $= b$ in.

If $\theta° < 90°$, we have by previous definitions,

$$\sin\theta° = \frac{b}{r}, \quad \cos\theta° = \frac{a}{r}, \quad \tan\theta° = \frac{b}{a}.$$

Now the previous definitions apply only to acute angles. For angles greater than 90°, *new definitions are necessary.*

The fact that the coordinates of P are $(+a, +b)$ suggests the form these new definitions should take.

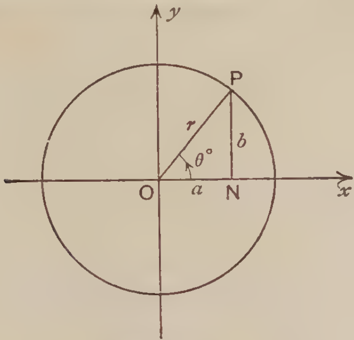


FIG. 159.

If $\theta° > 90°$, we **define** the trigonometrical ratios of θ as follows:

$$\sin\theta° = \frac{\text{y-coordinate of P}}{\text{r}}, \quad \cos\theta° = \frac{\text{x-coordinate of P}}{\text{r}},$$

$$\tan\theta° = \frac{\text{y-coordinate of P}}{\text{x-coordinate of P}}.$$

This definition evidently includes the original definition as a particular case and extends it.

(i) Suppose $\angle x$OP is obtuse. *i.e.* $180° > \theta° > 90°$, see Fig. 160.

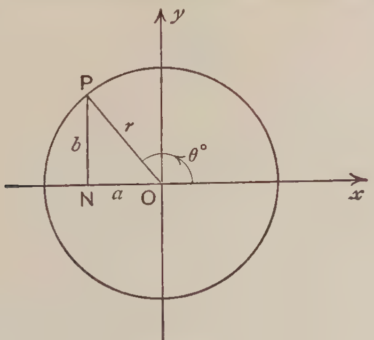


FIG. 160.

As before, let ON $=a$ inches, NP $=b$ inches.

Then the *coordinates* of P are $(-a, +b)$;

$\therefore$ by definition,
$$\sin\theta^\circ = \frac{+b}{r} = +\frac{b}{r},$$
$$\cos\theta^\circ = \frac{-a}{r} = -\frac{a}{r},$$
$$\tan\theta^\circ = \frac{+b}{-a} = -\frac{b}{a}.$$

$\therefore$ if θ° is obtuse, $\sin\theta^\circ$ is *positive*, but $\cos\theta^\circ$ and $\tan\theta^\circ$ are each *negative*.

Further, in Fig. 160, $\angle\mathrm{PON} = 180^\circ - \theta^\circ$ and is acute.

$$\therefore \sin\theta^\circ = \frac{b}{r} = \sin\mathrm{PON} = \sin(180^\circ - \theta^\circ),$$
$$\cos\theta^\circ = -\frac{a}{r} = -\cos\mathrm{PON} = -\cos(180^\circ - \theta^\circ),$$
$$\tan\theta^\circ = -\frac{b}{a} = -\tan\mathrm{PON} = -\tan(180^\circ - \theta^\circ).$$

For example,

$$\sin 138^\circ = \sin(180^\circ - 138^\circ) = \sin 42^\circ = 0{\cdot}6691,$$
and $$\cos 138^\circ = -\cos(180^\circ - 138^\circ) = -\cos 42^\circ = -0{\cdot}7431,$$
and $$\tan 138^\circ = -\tan(180^\circ - 138^\circ) = -\tan 42^\circ = -0{\cdot}9004.$$

(ii) Suppose $270^\circ > \theta^\circ > 180^\circ$, see Fig. 161.

As before, let ON $=a$ inches, NP $=b$ inches.

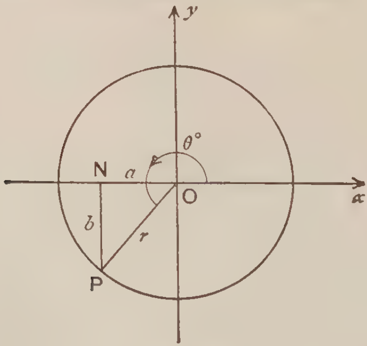


FIG. 161.

Then the *coordinates* of P are $(-a, -b)$;

$\therefore$ by definition, $\sin\theta^\circ = \frac{-b}{r} = -\frac{b}{r}$,

$$\cos\theta^\circ = \frac{-a}{r} = -\frac{a}{r},$$

$$\tan\theta^\circ = \frac{-b}{-a} = +\frac{b}{a}.$$

$\therefore$ if $270^\circ > \theta^\circ > 180^\circ$, $\tan\theta^\circ$ is *positive*, but $\sin\theta^\circ$ and $\cos\theta^\circ$ are each *negative*.

(iii) Suppose $360^\circ > \theta^\circ > 270^\circ$, see Fig. 162.

As before, let ON $= a$ inches, NP $= b$ inches.

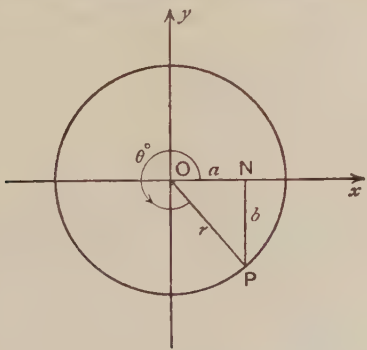


FIG. 162.

Then the *coordinates* of P are $(+a, -b)$;

$\therefore$ by definition, $\sin\theta^\circ = \frac{-b}{r} = -\frac{b}{r}$,

$$\cos\theta^\circ = \frac{+a}{r} = +\frac{a}{r},$$

$$\tan\theta^\circ = \frac{-b}{+a} = -\frac{b}{a}.$$

$\therefore$ if $360^\circ > \theta^\circ > 270^\circ$, $\cos\theta^\circ$ is positive, but $\sin\theta^\circ$ and $\tan\theta^\circ$ are each *negative*.

Numerical values of ratios. Suppose that in the four figures, Figs. 159-162, the four triangles ONP are congruent; then, *apart from sign*, the numerical values of any one trigonometrical ratio are all equal.

For example, suppose, in Fig. 159, $\theta° = 63°$; $\sin 63° = 0{\cdot}8910$.

Then in Fig. 160, $\theta° = 180° - 63° = 117°$,

$$\therefore \sin 117° = 0{\cdot}8910;$$

and in Fig. 161, $\theta° = 180° + 63° = 243°$,

$$\therefore \sin 243° = -0{\cdot}8910;$$

and in Fig. 162, $\theta° = 360° - 63° = 297°$,

$$\therefore \sin 297° = -0{\cdot}8910.$$

Similarly, since $\cos 63° = 0{\cdot}4540$, we have

$$\cos 117° = -0{\cdot}4540;\ \cos 243° = -0{\cdot}4540;\ \cos 297° = 0{\cdot}4540.$$

And, since $\tan 63° = 1{\cdot}9626$, we have

$$\tan 117° = -1{\cdot}9626;\ \tan 243 = 1{\cdot}9626;\ \tan 297° = -1{\cdot}9626.$$

We may state these results as follows:

The ratio of any angle xOP $= \theta°$ is **numerically equal** *to the* **same** *ratio of any angle whose sum with $\theta°$ or difference from $\theta°$ is a multiple of* 180°; *the* **sign** *of the value of the ratio is determined by the quadrant in which* OP *lies.*

FIG. 163.

The reader should determine this sign by drawing a figure as above. It may, however, be of interest to give a mnemonic; write the letters of the word CAST in the quadrants; these indicate which ratio is *positive* for the marked quadrant, **C**osine, **A**ll, **S**ine, **T**angent. Obviously all the ratios are positive in the first quadrant; this fixes the position of A; and the letters are written the same way round as OP rotates.

The following is a summary of the results established:

$$\mathbf{\sin (180° - \theta°) = \sin \theta°\ ;\ \sin (180° + \theta°) = -\sin \theta°\ ;}$$
$$\mathbf{\sin (360° - \theta°) = -\sin \theta°.}$$

$$\cos(180° - \theta°) = -\cos\theta°\ ;\ \cos(180° + \theta°) = -\cos\theta°\ ;$$
$$\cos(360° - \theta°) = \cos\theta°.$$
$$\tan(180° - \theta°) = -\tan\theta°\ ;\ \tan(180° + \theta°) = \tan\theta°\ ;$$
$$\tan(360° - \theta°) = -\tan\theta°.$$

EXERCISE VII. b.

The radius of each circle in Fig. 164 is 5 cm. ; ∠AOP, swept out anti-clockwise, equals $\theta°$. Write down the values of $\sin\theta°$, $\cos\theta°$, $\tan\theta°$ corresponding to the data in Nos. 1-9.

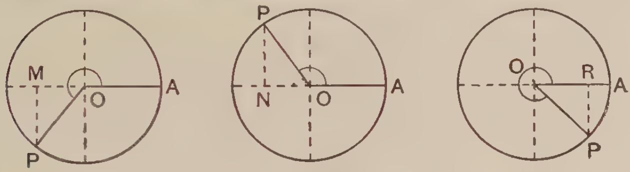


FIG. 164.

1. OM = 3 cm.	**2.** ON = 3 cm.	**3.** OR = 3 cm.
4. ON = 4 cm.	**5.** OR = 4 cm.	**6.** OM = 4 cm.
7. NP = 4 cm.	**8.** RP = 3 cm.	**9.** OM = 2 cm.

10. Draw on squared paper a circle of radius 1 inch, and, *by drawing the actual angles involved*, use it to find approximate values of

(i) $\sin 115°$; (ii) $\cos 115°$; (iii) $\cos 220°$; (iv) $\tan 220°$;
(v) $\sin 305°$; (vi) $\cos 305°$; (vii) $\cos 170°$; (viii) $\tan 140°$.

What can you say about θ in the following cases, Nos. 11-16 ?

11. $\sin\theta°$ is positive, $\cos\theta°$ is negative.

12. $\cos\theta°$ is positive, $\sin\theta°$ is negative.

13. $\tan\theta°$ and $\cos\theta°$ are both negative.

14. $\cos\theta°$ is negative, $\tan\theta°$ is positive.

15. $\cos\theta°$ is positive, $\sin\theta°$ is negative.

16. $\sin\theta°$ and $\tan\theta°$ are both negative.

Express each of the ratios in Nos. 17-28 as the ratio of an *acute* angle with the appropriate sign (thus : $\sin 290° = -\sin 70°$):

17. $\cos 200°$.	**18.** $\sin 170°$.	**19.** $\sin 340°$.	**20.** $\cos 280°$.
21. $\sin 190°$.	**22.** $\cos 165°$.	**23.** $\sin 260°$.	**24.** $\cos 250°$.
25. $\tan 145°$.	**26.** $\sin 95°$.	**27.** $\tan 230°$.	**28.** $\tan 325°$.

Construct on squared paper the following angles and measure them (two answers in each case):

29. $\cos^{-1}(-\frac{2}{5})$. **30.** $\sin^{-1}(-\frac{3}{4})$. **31.** $\tan^{-1}(-\frac{1}{2})$. **32.** $\tan^{-1}(\frac{3}{5})$.

Find from the tables the values of the following:

33. sin 160°. **34.** cos 195°. **35.** tan 300°. **36.** cos 155°.

37. tan 210°. **38.** sin 317°. **39.** cos 317°. **40.** sin 215°.

41. sin 123° 40′. **42.** cos 123° 40′.

43. tan 216° 25′. **44.** cos 308° 35′.

Example I. Draw with the same axes and scale the graphs of $\sin x°$ and $\cos x°$ for values of x from 0 to 360.

We have from the tables the following values:

x	0	45	90	135	180	225	270	315	360
$\sin x°$	0	0·71	1	0·71	0	−0·71	−1	−0·71	0
$\cos x°$	1	0·71	0	−0·71	−1	−0·71	0	0·71	1

In Figure 165, the graph of $\sin x°$ is represented by a continuous curve and the graph of $\cos x°$ by a broken curve.

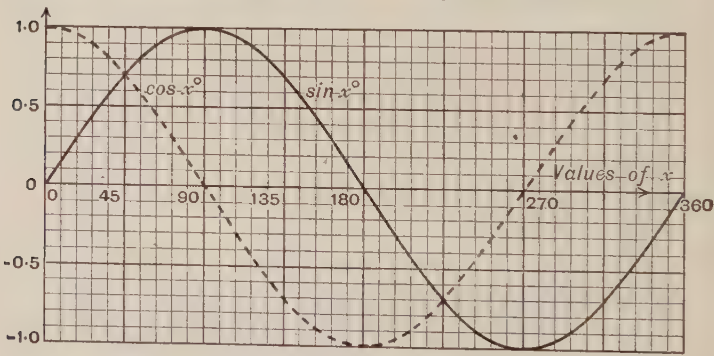


FIG. 165.

Note. 1. In order to draw a reasonably accurate graph, it is necessary to take a larger number of values of x than are

shown in the table above. (Compare the graph and table of values on p. 80.)

2. The definitions for sines and cosines of angles of any magnitude give smooth curves for each graph; further, if the graph of $\cos x°$ is moved in the direction of the x-axis through 90 units, it coincides with the graph of $\sin x°$; this is equivalent to saying that $\sin(x° + 90°) = \cos x°$.

3. For oral questions on this graph, see Ex. VII. c., Nos. 1-8.

The remaining trigonometrical ratios of θ, when $\theta° > 90°$, are defined in accordance with the definitions already given on p. 39.

Thus for *all values* of θ, we say that

$$\operatorname{cosec} \theta° = \frac{1}{\sin \theta°}; \quad \sec \theta° = \frac{1}{\cos \theta°}; \quad \cot \theta° = \frac{1}{\tan \theta°}.$$

Hence $\quad \operatorname{cosec} 117° = \dfrac{1}{\sin 117°} = \dfrac{1}{\sin 63°} = \operatorname{cosec} 63°,$

$$\sec 117° = \frac{1}{\cos 117°} = \frac{1}{-\cos 63°} = -\sec 63°,$$

$$\cot 117° = \frac{1}{\tan 117°} = \frac{1}{-\tan 63°} = -\cot 63°.$$

And, in general,

$$\mathbf{cosec}\ \theta° = \mathbf{cosec}\ (\mathbf{180}° - \theta°); \quad \mathbf{sec}\ \theta° = -\mathbf{sec}\ (\mathbf{180}° - \theta°);$$
$$\mathbf{cot}\ \theta° = -\mathbf{cot}\ (\mathbf{180}° - \theta°).$$

EXERCISE VII. c.

Use the graphs in Fig. 165 for Nos. 1-8.

1. What are the values of $\cos 27°$, $\cos 153°$, $\cos 207°$, $\cos 333°$; $\sin 117°$, $\sin 243°$, $\sin 297°$?

2. What is x if (i) $\cos x° = -0{\cdot}3$; (ii) $\sin x° = -0{\cdot}3$?

3. What is x if (i) $\sin x° = 0{\cdot}8$; (ii) $\cos x° = 0{\cdot}8$?

4. For what range of values of x between 0 and 360 is (i) $\sin x°$ negative; (ii) $\cos x°$ negative?

5. What can you say about x if (i) $\sin x^\circ > 0{\cdot}4$; (ii) $\sin x^\circ < -0{\cdot}4$?

6. What can you say about x if (i) $\cos x^\circ > 0{\cdot}4$; (ii) $\cos x^\circ < -0{\cdot}4$?

7. For what values of x is $\sin x^\circ = \cos x^\circ$?

8. Make a rough copy of Fig. 165, and show how you think the graphs continue for values of x beyond 360.

What would you expect the graphs to be for negative values of x ?

9. *Sketch* the graphs of $\sin (2x^\circ)$ and $\cos (2x^\circ)$ for values of x from 0 to 180.

Use the tables to find two solutions of each of the following equations:

10. $\cos \theta^\circ = 0{\cdot}5$. **11.** $\sin \theta^\circ = 0{\cdot}342$. **12.** $\tan \theta^\circ = 1{\cdot}6$.

13. $\sin \theta^\circ = -0{\cdot}766$. **14.** $\cos \theta^\circ = -0{\cdot}454$. **15.** $\tan \theta^\circ = -0{\cdot}404$.

Find from the tables the values of the following:

16. cosec 200°. **17.** sec 310°. **18.** cot 165°. **19.** sec 140°.

20. cot 265°. **21.** cosec 100°. **22.** sec 230°. **23.** cot 310°.

Use the tables to solve the following equations:

24. $\cot x^\circ = \frac{1}{2}$. **25.** $\operatorname{cosec} x^\circ = 2{\cdot}5$. **26.** $\sec x^\circ = -2{\cdot}4$.

27. $\operatorname{cosec} x^\circ = -2{\cdot}4$. **28.** $\tan^2 x^\circ = 4$. **29.** $\sec^2 x^\circ = 3$.

30. Draw on the same figure

(i) the graph of $\tan x^\circ$ for values of x from 0 to 75, and from 105 to 255, and from 285 to 360;

(ii) the graph of $\cot x^\circ$ for values of x from 15 to 165, and from 195 to 345.

31. Draw on the same figure

(i) the graph of $\operatorname{cosec} x^\circ$ for values of x, from 15 to 165, and from 195 to 345;

(ii) the graph of $\sec x^\circ$ for values of x from 0 to 75, and from 105 to 255, and from 285 to 360.

32. Find x if

(i) $\sin x^\circ = 0{\cdot}6$ and $\cos x^\circ = -0{\cdot}8$;

(ii) $\cos x^\circ = 0{\cdot}6$ and $\sin x^\circ = -0{\cdot}8$;

(iii) $\cos x^\circ = 0{\cdot}8$ and $\tan x^\circ = -0{\cdot}75$.

Simplify the following:

33. cosec $(180° - \theta)$. **34.** sec $(180° + \theta)$. **35.** cot $(360° - \theta)$.

36. sec $(360° - \theta)$. **37.** cosec $(180° + \theta)$. **38.** cot $(180° + \theta)$.

39. The angles of a triangle are A°, B°, C°; express sin (B° + C°) and cos (B° + C°) in terms of A.

Generalisations. The introduction of ratios of angles of any magnitude enables many results to be stated in a more general form than would otherwise be possible. The definitions on p. 99 have been so chosen that in general formulae which are established for acute angles hold equally for angles of any magnitude.

Example 11. A man walks 3 miles along a straight road whose true bearing is $\theta°$. How far (i) north, (ii) east is he of his starting point? Consider also the special cases when the true bearings are 160 and 245°.

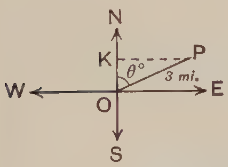


FIG. 166.

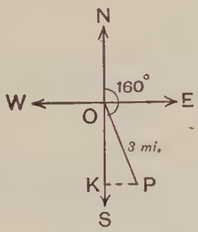


FIG. 167.

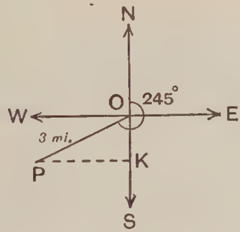


FIG. 168.

In Fig. 166, where $\angle NOP = \theta° < 90°$, we see that P is $3 \cos \theta°$ miles North of O and $3 \sin \theta°$ miles East of O. These statements remain true whatever size the angle $\theta°$ is.

Thus, in Fig. 167, $\theta° = 160°$, and these results become $3 \cos 160°$ miles North of O and $3 \sin 160°$ miles East of O. But, since $\angle KOP = 180° - 160° = 20°$, it is clear that P is $3 \cos 20°$ miles *South* of O and $3 \sin 20°$ miles East of O; or we may say that P is $-3 \cos 20°$ miles *North* of O.

The two sets of results are therefore consistent if

$$\cos 160° = -\cos 20° \quad \text{and} \quad \sin 160° = +\sin 20°;$$

and we have already seen (pp. 102-103) that this is so.

Again, in Fig. 168, $\theta° = 245°$, and the general results become $3 \cos 245°$ miles North of O and $3 \sin 245°$ miles East of O. But, since $\angle \text{KOP} = 245° - 180° = 65°$, it is clear that P is $3 \cos 65°$ miles *South* of O and $3 \sin 65°$ miles *West* of O; or we may say that P is $-3 \cos 65°$ miles *North* of O and $-3 \sin 65°$ miles *East* of O.

The two sets of results are therefore consistent if

$$\cos 245° = -\cos 65° \quad \text{and} \quad \sin 245° = -\sin 65°;$$

and we have already seen (pp. 102-103) that this is so.

Example III. Find the area of $\triangle$ABC in terms of two sides and the included angle, say, b, c, A.

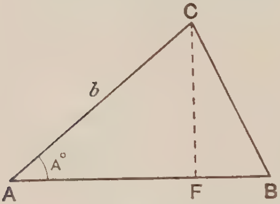


FIG. 169.

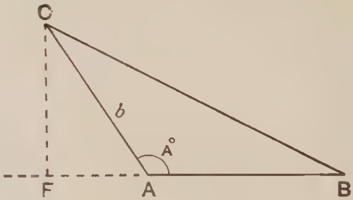


FIG. 170.

Draw CF perpendicular to AB or AB produced.

$$\text{The area of } \triangle \text{ABC} = \tfrac{1}{2}\text{AB} \,.\, \text{FC} = \tfrac{1}{2}c \,.\, \text{FC}.$$

In Fig. 169, $\quad \text{FC} = b \sin \text{A}.$

In Fig. 170, $\quad \text{FC} = b \sin \text{F}\hat{\text{A}}\text{C} = b \sin (180° - \text{A}) = b \sin \text{A};$

$$\therefore \text{ in each case, area of } \triangle \text{ABC} = \tfrac{1}{2}c \,.\, b \sin \text{A}$$
$$= \tfrac{1}{2}\text{bc} \sin \text{A}.$$

Note. This formula for the area of a triangle is therefore the same whether the included angle is acute or obtuse; obviously this is a great convenience. It is a direct consequence of the definition chosen above for the sine of an obtuse angle.

This result was first given by Snell (1627), Professor of Mathematics at Leyden.

Example IV. With the usual notation for the $\triangle ABC$, prove that

$$a = b \cos C + c \cos B.$$

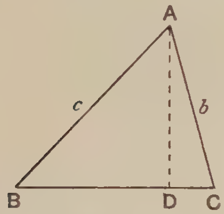


FIG. 171.

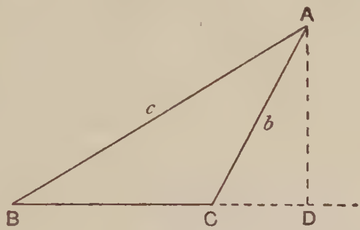


FIG. 172.

Draw AD perpendicular to BC or BC produced.

In Fig. 171, $a = BC = BD + DC = c \cos B + b \cos C$.

In Fig. 172, $a = BC = BD - CD = c \cos B - b \cos D\hat{C}A$

$$= c \cos B - b \cos (180° - C);$$

but $\cos (180° - C) = -\cos C$;

$$\therefore \; a = c \cos B + b \cos C.$$

Note. The relation proved above is therefore true alike for acute-angled and obtuse-angled triangles; obviously this is a great convenience. It is a direct consequence of the definition chosen above for the cosine of an obtuse angle.

EXERCISE VII. d.

1. A rod OP, 5 ft. long, rotates about O through 10° per second in a vertical plane, anti-clockwise, starting from the horizontal OA.

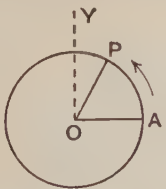


FIG. 173.

Show that after t seconds, the height of P above OA is $5 \sin (10t°)$ feet. Evaluate this expression for $t = 3$, 15, 21, 33, and show roughly on a figure the position corresponding to each case.

2. With the data of No. 1, find the height of P above OA after 5 seconds and the time when it is next at the same height.

3. With the data of No. 1, find how far P is to the right of the vertical line OY after t seconds. Evaluate this expression for $t=3$, 15, 21, 33, and show roughly on a figure the corresponding positions.

4. With the data of No. 1, find after what times P will be (i) 2 feet to the right of the vertical line OY, (ii) 2 feet to the left of OY.

5. In a certain tidal channel, in t hours the water rises $12 \sin\left(\frac{144t^\circ}{5}\right)$ feet above mean level. If it is mean level at 2 a.m., find the height above mean level at 7 a.m., 12 noon, 5 p.m., 10 p.m. on the same day.

6. With the data of No. 5, find the times of (i) high-water, (ii) low-water during the 24 hours after 2 a.m.

7. A buoy in the sea is rising and falling vertically with the waves; its height above the mean level after t seconds from the time when first observed is $5 \cos (30t^\circ)$ feet. Find this height for $t=2$, 4, 6, 8, 10, 12. Through what distance does it oscillate? What is the time of one complete oscillation?

8. There is a steady wind of 24 m.p.h. blowing from θ° East of North; an aeroplane starts from O and heads due South; if there were no wind it would be travelling at 90 m.p.h.; but owing to the

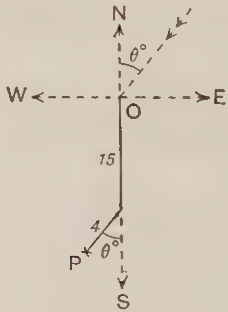


FIG. 174.

wind its position after 10 minutes is shown by the point P in Fig. 174. How far, in terms of θ, is P (i) south, (ii) west of O? Evaluate these expressions if θ° equals (a) 20°, (b) 160°, (c) 0°, (d) 180°, (e) 360°, (f) 200°, (g) 340°, and illustrate your answers by rough figures, showing the various positions.

9. The legs of a compass, each l inches long, are opened to an angle $2\alpha°$. Show that the distance between the points is $2l \sin \alpha°$ inches. Is this true if $\alpha° > 90°$? What happens if $\alpha = 180°$?

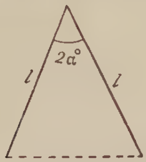


FIG. 175.

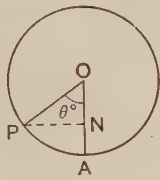


FIG. 176.

10. A rod OP, l inches long, swings about O in a vertical plane; PN is perpendicular to the vertical OA; if $\angle \text{AOP} = \theta°$, show that $\text{AN} = l(1 - \cos \theta°)$ inches. Is this still true if $\theta° > 90°$? What happens if (i) $\theta = 180°$, (ii) $\theta = 270°$, (iii) $\theta = 360°$?

11. Prove that the area of the parallelogram in Fig. 177 is $ab \sin \theta$.

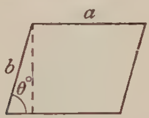


FIG. 177.

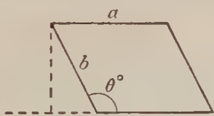


FIG. 178.

Is the same formula true for the parallelogram in Fig. 178?

12. In Fig. 179, express the length of AD in two different ways, and

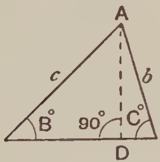


FIG. 179.

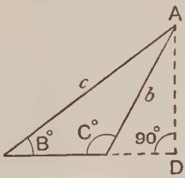


FIG. 180.

so prove that $\dfrac{b}{\sin \text{B}} = \dfrac{c}{\sin \text{C}}$. Is this result also true for Fig. 180?

13. It has been proved (p. 55) that, if θ is any acute angle, then $\sin^2\theta + \cos^2\theta = 1$ and $\dfrac{\sin \theta}{\cos \theta} = \tan \theta$. Prove from the definitions on p. 99 that these formulae are always true.

14. In Fig. 181, a, b, c, d are the feet of the perpendiculars from A, B, C, D to a given line OX; AB, BC, CD make angles α, β, γ with

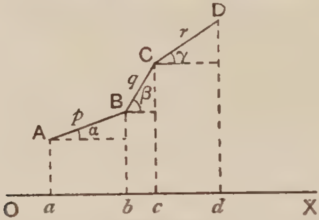


FIG. 181.

OX, measured anti-clockwise from OX; $AB = p$, $BC = q$, $CD = r$.

Prove that $ad = p \cos \alpha + q \cos \beta + r \cos \gamma$; and find an expression for the height of D above A, if OX is regarded as horizontal and Aa, Bb, etc., as vertical.

Evaluate these expressions, taking $p = 1$, $q = 1$, $r = 2$, in the following cases, illustrating each answer by a rough figure:

(i) $\alpha = 30°$, $\beta = 110°$, $\gamma = 200°$;
(ii) $\alpha = 140°$, $\beta = 50°$, $\gamma = 310°$;
(iii) $\alpha = 110°$, $\beta = 320°$, $\gamma = 200°$.

15. With the data of No. 14, show that the angle $\theta°$ which the line joining A to C makes with OX is given by $\tan \theta = \dfrac{p \sin \alpha + q \sin \beta}{p \cos \alpha + q \cos \beta}$.
Taking $p = 1$, $q = 1$, find θ in the following cases, illustrating each answer by a rough figure:

(i) $\alpha = 50°$, $\beta = 130°$; (ii) $\alpha = 300°$, $\beta = 20°$;
(iii) $\alpha = 110°$, $\beta = 10°$; (iv) $\alpha = 220°$, $\beta = 100°$.

16. (i) If in Fig. 182, PQ = QR, prove that $\cot \beta = \frac{1}{2}(\cot \alpha + \cot \gamma)$.

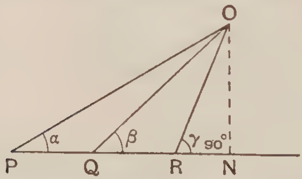


FIG. 182.

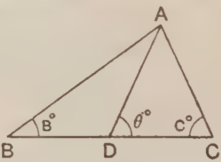


FIG. 183.

(ii) If in Fig. 183, AD is a median of $\triangle$ABC, use (i) to express $\cot \theta$ in terms of B, C; and evaluate θ in the following cases, illustrating each by a rough figure:

(1) $B = 35°$, $C = 50°$; (2) $B = 35°$, $C = 105°$; (3) $B = 50°$, $C = 35°$.

CHAPTER VIII.

USE OF LOGARITHM TABLES.

THE numerical work necessary in applications of Trigonometry can generally be shortened by using logarithms, and, to save time, tables of logarithms of each trigonometrical ratio have been constructed. For example, we find on one page that $\sin 50° = 0{\cdot}7660$ and on another page that $\log 0{\cdot}7660 = \bar{1}{\cdot}8842$; therefore $\log \sin 50° = \bar{1}{\cdot}8842$; but if we use the table of log-sines we obtain this result by a single reading.

It is not always easy to fix the characteristic by common-sense; it is therefore always printed, but only at the beginning of each line;

$$e.g.\ \log \tan 10°\ 30' = 1{\cdot}2680; \quad \log \tan 84°\ 30' = 1{\cdot}0164.$$

As in the case of the natural ratios, the figures in the difference columns must be *subtracted* for log cosines, log cosecants and log cotangents.

The figures in the difference columns can only be *average differences*, and, usually, sufficient accuracy is secured by taking the average over an interval of one degree. When, however, this introduces an appreciable error, the difference columns in the tables at the end of the book give *average differences for one minute*, calculated over "12 minute intervals."

e.g. To find $\log \tan 84°\ 56'$ and $\log \tan 84°\ 58'$.

For the interval 48′ to 60′, the difference for 1′ is given as 14,

$\therefore$ the difference for 2′ is 28, *i.e.* ·0028.

$$\log \tan 84^\circ\ 54' \simeq 1{\cdot}0494,$$

$$\therefore \log \tan 84^\circ\ 56' \simeq 1{\cdot}0494 + {\cdot}0028 = 1{\cdot}0522.$$

$$\log \tan 85^\circ \simeq 1{\cdot}0580,$$

$$\therefore \log \tan 84^\circ\ 58' \simeq 1{\cdot}0580 - {\cdot}0028 = 1{\cdot}0552.$$

Note. The difference correction is applied to the *nearest* angle given in the tables.

Example. It is proved in Chapter IX. (see p. 119) that, in any triangle ABC, $\dfrac{a}{\sin \mathsf{A}} = \dfrac{b}{\sin \mathsf{B}} = \dfrac{c}{\sin \mathsf{C}}$.

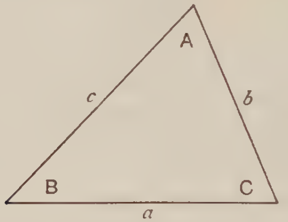


FIG. 184.

(i) Given $b = 4{\cdot}618$, $\mathsf{B} = 41^\circ\ 29'$, $\mathsf{C} = 57^\circ\ 37'$, find c.

(ii) Given $b = 67{\cdot}24$, $c = 89{\cdot}69$, $\mathsf{C} = 62^\circ\ 46'$, find B.

(i) $$\frac{c}{\sin 57^\circ\ 37'} = \frac{4{\cdot}618}{\sin 41^\circ 29'};$$

$$\therefore c = \frac{4{\cdot}618 \sin 57^\circ\ 37'}{\sin 41^\circ\ 29'}$$

$$= 5{\cdot}887 \simeq \mathbf{5{\cdot}89}.$$

(ii) $$\frac{\sin \mathsf{B}}{67{\cdot}24} = \frac{\sin 62^\circ\ 46'}{89{\cdot}69}$$

$$\therefore \sin \mathsf{B} = \frac{67{\cdot}24 \sin 62^\circ\ 46'}{89{\cdot}69};$$

$$\therefore \mathsf{B} = \mathbf{41^\circ\ 50'}.$$

	Number (numerator)	log +	log −	Number (denominator)
(i)	4·618 sin 57° 37′	0·6644 $\bar{1}$·9266 0·5910 $\bar{1}$·8211 0·7699	$\bar{1}$·8211	sin 41° 29′
(ii)	67·24 sin 62° 46′	1·8277 $\bar{1}$·9490 1·7767 1·9527 $\bar{1}$·8240	1·9527	89·68

Note. (i) The working should at first be set out in the way shown above. but after a reasonable standard of accuracy has

been attained the side-columns which show the numbers may be omitted. Logarithms should never be allowed to appear in the middle of the page unless expressed either in the index form or in some other unambiguous fashion.

(ii) When using *four-figure* tables, results should as a rule be given correct to 3 significant figures; although this applies also to angles, it is convenient to give angles to the nearest minute, but it should be realised that the fourth figure is not reliable.

EXERCISE VIII.

1. Look up sin 55° and then look up the logarithm of this number. What does the "Log-Sine" table give for log sin 55°?

Repeat this process for (i) cos 37° 24′; (ii) tan 48° 30′; (iii) cot 85° 18′; (iv) sec 76° 57′; (v) cosec 23° 29′.

2. Find from the tables the values of the following:

(i) log sin 17° 36′; (ii) log cos 63° 32′; (iii) log cot 65° 26′; (iv) log tan 17° 16′; (v) log cosec 5° 36′; (vi) log sec 84° 24′.

Why are (v) and (vi) equal?

3. Find x, given that

(i) $\log\sin x^\circ = \bar{1}\cdot9023$; (ii) $\log\cos x^\circ = \bar{1}\cdot9435$;
(iii) $\log\tan x^\circ = 0\cdot478$; (iv) $\log\cot x^\circ = \bar{1}\cdot9006$;
(v) $\log\sec x^\circ = 0\cdot0573$; (vi) $\log\operatorname{cosec} x^\circ = 1\cdot003$;
(vii) $\log\sin x^\circ = \bar{1}\cdot5491$; (viii) $\log\tan x^\circ = \bar{1}\cdot9$;
(ix) $\log\cos x^\circ = \bar{1}\cdot5819$; (x) $\log\cot x^\circ = \bar{1}\cdot55$;
(xi) $\log\operatorname{cosec} x^\circ = 0\cdot5691$; (xii) $\log\sec x^\circ = 0\cdot6005$.

Evaluate the following, Nos. 4-15.

4. $\dfrac{\sin 72^\circ}{\sin 51^\circ}$. **5.** $\sin 18^\circ \cos 18^\circ$. **6.** $\cos^2 37^\circ 18'$. **7.** $\dfrac{\sin 37^\circ 15'}{\tan 49^\circ 24'}$.

8. $\dfrac{3\cdot42 \sin 33^\circ 15'}{\sin 41^\circ 18'}$. **9.** $\dfrac{11\cdot5 \tan 27^\circ 11'}{\tan 73^\circ 15'}$. **10.** $\dfrac{53\cdot8 \cos 17^\circ 38'}{\cos 54^\circ 20'}$.

11. $\cos^3 54^\circ 40'$. **12.** $\dfrac{\tan 32^\circ 20'}{3\cdot64 \tan 22^\circ 17'}$. **13.** $\dfrac{\sin^2 55^\circ 25'}{\cos^2 37^\circ 32'}$.

14. $\dfrac{18\cdot72 \sin 57^\circ}{203 \sin 4^\circ}$. **15.** $33\cdot62 \operatorname{cosec} 18^\circ 11' \sec 39^\circ 16'$.

Find a value of θ satisfying the following equations, Nos. 16-24.

16. $\sin\theta^\circ = \dfrac{17\cdot34}{27\cdot92}$. **17.** $\cos\theta^\circ = \dfrac{137}{249}$. **18.** $\cot\theta^\circ = \dfrac{0\cdot8651}{1\cdot907}$.

19. $\sin\theta^\circ = \dfrac{11{\cdot}3 \sin 17^\circ 45'}{10{\cdot}87}$.

20. $\sin\theta^\circ = \dfrac{23{\cdot}71 \sin 69^\circ 17'}{27{\cdot}18}$.

21. $\cos\theta^\circ = \sqrt{\left(\dfrac{18{\cdot}62 \times 3{\cdot}49}{82{\cdot}71}\right)}$.

22. $\tan\theta^\circ = \dfrac{3{\cdot}72 \tan 17^\circ 52'}{2{\cdot}936}$.

23. $\sec\theta^\circ = \dfrac{8{\cdot}073 \times 2{\cdot}497}{17{\cdot}62}$.

24. $\cos\theta^\circ = \dfrac{1{\cdot}86 \cos 63^\circ 10'}{2{\cdot}071}$.

25. If $\cos\phi = \sin\alpha \sin\beta$, find ϕ given $\alpha = 14^\circ 50'$, $\beta = 67^\circ 25'$.

26. If $r = 4\mathrm{R} \sin\dfrac{\mathrm{A}}{2} \sin\dfrac{\mathrm{B}}{2} \sin\dfrac{\mathrm{C}}{2}$; find r, given that $\mathrm{R} = 3{\cdot}25$, $\mathrm{A} = 57^\circ$, $\mathrm{B} = 49^\circ$, $\mathrm{C} = 74^\circ$.

27. Evaluate $\dfrac{c \sin\alpha \sin\beta}{\cos\alpha + \cos\beta}$, when $c = 147$, $\alpha = 42^\circ\ 17'$, $\beta = 51^\circ 42'$.

28. From the formula for the volume V cu. in. of a cone

$$\mathrm{V} = \tfrac{1}{3}\pi h^3 \tan^2\alpha,$$

where α is the semi-vertical angle and h in. is the height, find V, given $h = 11$, $\alpha = 14^\circ$.

29. With the notation of No. 28, find α, given $\mathrm{V} = 60$, $h = 6$.

30. In any $\triangle\mathrm{ABC}$, $\dfrac{\tan\dfrac{\mathrm{B}-\mathrm{C}}{2}}{\tan\dfrac{\mathrm{B}+\mathrm{C}}{2}} = \dfrac{b-c}{b+c}$; Find B, C given that $b = 812$, $c = 639$, $\mathrm{A} = 37^\circ$.

31. Find A from the formula $\sin\dfrac{\mathrm{A}}{2} = \sqrt{\left[\dfrac{(s-b)(s-c)}{bc}\right]}$, where $b = 43$, $c = 37$, $s = 59$.

32. Find θ from the formula $\tan\dfrac{\theta}{2} = \tan\dfrac{\alpha}{2} \cdot \sqrt{\left(\dfrac{1+e}{1-e}\right)}$, where $e = 0{\cdot}43$ and $\alpha = 23^\circ 20'$.

33. The volume of a triangular pyramid is given by the formula

$$\mathrm{V} = \tfrac{1}{3}abc\sqrt{[\sin\sigma \sin(\sigma-\alpha) \sin(\sigma-\beta) \sin(\sigma-\gamma)]};$$

find V, given $a = 3{\cdot}7$, $b = 4{\cdot}4$, $c = 2{\cdot}9$, $\alpha = 62^\circ$, $\beta = 51^\circ$, $\gamma = 73^\circ$,

$$\sigma = \tfrac{1}{2}(\alpha + \beta + \gamma).$$

34. Find A from the formula for a spherical triangle:

$$\sin\frac{\mathrm{A}}{2} = \sqrt{\left[\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}\right]},$$

where $a = 59^\circ 20'$, $b = 39^\circ 40'$, $c = 47^\circ 33'$ and $s = \tfrac{1}{2}(a+b+c)$.

35. From the formula $\cos a = \cos b \cos c + \sin b \sin c \cos\mathrm{A}$, find a, given $b = 73^\circ 55'$, $c = 61^\circ 20'$, $\mathrm{A} = 22^\circ 30'$.

CHAPTER IX.

SOLUTION OF TRIANGLES.

In Elementary Geometry, the tests for congruent triangles are ascertained by enquiring what various sets of data are necessary and sufficient for copying a triangle. This work should be revised orally before proceeding to the formal methods of solution.

EXERCISE IX. a. (Oral.)

Is it possible to draw several, one or no triangle subject to the

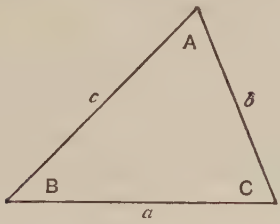


FIG. 184.

following conditions? Draw *rough* figures in each case.

1. $b=1$, $c=2$, $A=52°$.
2. $b=1$, $c=2$, $a=4$.
3. $b=3$, $c=4$, $a=5$, $A=90°$.
4. $a=3$, $c=5$, $B=127°$.
5. $a=5$, $b=7$, $c=9$.
6. $A=40°$, $B=60°$, $C=80°$.
7. $b=8$, $c=7$, $C=30°$.
8. $b=8$, $c=9$, $C=30°$.
9. $b=8$, $c=1$, $C=30°$.
10. $b=8$, $c=4$, $C=30°$.
11. $A=45°$, $B=65°$, $C=60°$.
12. $a=2$, $B=100°$, $A=44°$.
13. $a=2$, $B=110°$, $C=70°$.
14. $c=10$, $a=8$, $A=40°$.
15. What relation connects A, B, C?

16. Are the values of a, b, c subject to any conditions? If so, what are they?

17. Name one set of three measurements which fixes a triangle uniquely. How many other such sets of a different kind are there, and what are they?

18. Invent a numerical example in which a, b, A are all given and in consequence of which it is found that it is possible to draw (i) two triangles of different size, (ii) only one triangle, (iii) no triangle at all, satisfying the data. Illustrate by rough figures.

Noting that there are six fundamental elements of a triangle, 3 sides and 3 angles, we may summarise the ideas of the last exercise as follows:

(1) The triangle is determined *uniquely* if we are given (i) the 3 sides, (ii) 2 sides and the included angle, (iii) 1 side and 2 angles.

Note. In (i) any side must be less than the sum of the other two; and in (iii) the sum of the two angles must be less than 180°.

(2) There may be two, one or no possible solution, if we are given two sides and the angle opposite one of them. [Ex. IX. a. Nos. 7-10.]

(3) The triangle cannot be determined unless the data include the length of at least one side.

Since the necessary data include 3 elements, one of which at least is a length, the remaining elements may be calculated from formulae connecting together *four* of the six elements; and two at least of these four must be lengths of sides.

The sine formula.

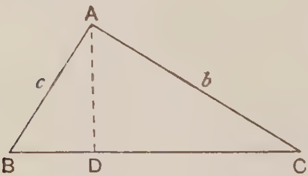


FIG. 185.

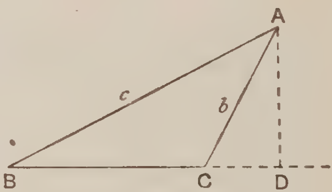


FIG. 186.

In any triangle ABC,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Draw AD perpendicular to BC, produced if necessary.

In Fig. 185, $AD = c \sin B$ and $AD = b \sin C$.

In Fig. 186, $AD = c \sin B$ and $AD = b \sin (180° - C) = b \sin C$.

$\therefore$ in each case, $b \sin C = c \sin B$;

$$\therefore \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Similarly, by drawing a perpendicular from C to AB, produced if necessary, we can prove that $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$;

$$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Note. **Never** apply the sine formula to a right-angled triangle. It is of course true; but it is mere waste of time to use it, when all that is necessary is the use of the definition of a sine or cosine.

A straightforward illustrative example on the uses of the sine formula is given on p. 114.

The ambiguous case. To construct the triangle ABC, given a, b and the angle A, *which is acute.*

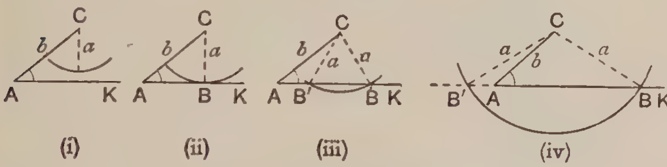


FIG. 187.

Draw the angle CAK equal to the given $\angle A$ and make AC equal to b. With centre C and radius a describe a circle. There are various possibilities:

(1) This circle may not cut AK at all, Fig. (i); then *no triangle can be drawn* to fit the data.

(2) This circle may *touch* AK at B, Fig. (ii); then *there is one triangle* ABC *and it is right angled at* B.

(3) This circle may cut AK at points B, B′ on the same side of A, Fig. (iii); then *there are two different triangles* ABC *and* AB′C, which fit the data.

(4) This circle may cut AK at points B, B′ on opposite sides of A, Fig. (iv); then $\triangle$ABC *fits the data and* $\triangle$AB′C *does not.*

We may state these results as follows, using the fact that the length of the perpendicular from C to AK equals $b \sin A$:

(1) If $a < b \sin A$, there is no solution, Fig. 187 (i).

(2) If $a = b \sin A$, one triangle exists and it is right angled, Fig. 187 (ii).

(3) If $b > a > b \sin A$, there are two distinct solutions, Fig. 187 (iii).

(4) If $a > b$, there is one and only one solution, Fig. 187 (iv).

Note. (i) The case of two distinct solutions arises only when the given angle is opposite the *shorter* of the two given sides.

(ii) If there are two distinct solutions, Fig. 187 (iii), the angles ABC, AB′C are supplementary; for CB = CB′;

$$\therefore \angle CBA = \angle CBB' = \angle CB'B = 180° - \angle CB'A.$$

(iii) If the given angle A is *obtuse*, there cannot be more than one solution, and there may not be any.

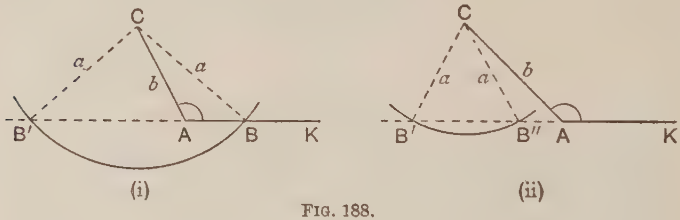


FIG. 188.

In Fig. 188 (i), $a > b$; one solution exists; $\triangle$ABC fits the data, but $\triangle$AB′C does not.

In Fig. 188 (ii), $a < b$; no solution exists; neither $\triangle$AB′C nor $\triangle$AB″C fit the data.

(iv) If the given angle A is a right angle, there is one solution if $a > b$ and no solution if $a < b$.

Example I. Solve $\triangle$ABC, given $a=8$, $b=10$, $\mathrm{A}=40°$.

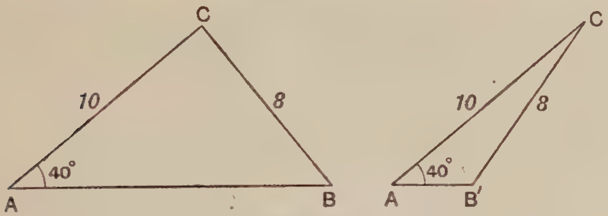


FIG. 189.

From the formula $\dfrac{\sin \mathrm{A}}{a}=\dfrac{\sin \mathrm{B}}{b}$, we have

$$\frac{\sin \mathrm{B}}{10}=\frac{\sin 40°}{8}; \quad \therefore \ \sin \mathrm{B}=\frac{10 \sin 40°}{8}=\frac{6{\cdot}428}{8}={\cdot}8035.$$

From the tables, $\sin 53° \, 28' = {\cdot}8035$; $\therefore \ \sin 126° \, 32' = {\cdot}8035$;

$$\therefore \ \mathrm{B}=53° \, 28' \text{ or } 126° \, 32'.$$

In Fig. 189, $\angle \mathrm{ABC} = \mathbf{53° \, 28'}$ and $\angle \mathrm{AB'C} = \mathbf{126° \, 32'}$.

$$\therefore \ \angle \mathrm{ACB} = 180° - 40° - 53° \, 28' = \mathbf{86° \, 32'}$$

and $$\angle \mathrm{ACB'} = 180° - 40° - 126° \, 32' = \mathbf{13° \, 28'}.$$

$$\therefore \ \frac{\mathrm{AB}}{\sin 86° \, 32'}=\frac{8}{\sin 40°};$$

$$\therefore \ \mathrm{AB}=\frac{8 \sin 86° \, 32'}{\sin 40°}=\mathbf{12{\cdot}4}.$$

0·9031	$\bar{1}$·8081
$\bar{1}$·9992	
0·9023	
$\bar{1}$·8081	
1·0942	

And $$\frac{\mathrm{AB'}}{\sin 13° \, 28'}=\frac{8}{\sin 40°};$$

$$\therefore \ \mathrm{AB'}=\frac{8 \sin 13° \, 28'}{\sin 40°}=\mathbf{2{\cdot}90}.$$

0·9031	$\bar{1}$·8081
$\bar{1}$·3671	
0·2702	
1·8081	
0·4621	

Note. There are two solutions, because the given angle is opposite the shorter of the two given sides.

Example II. In the $\triangle ABC$, given $a=12$, $b=10$, $A=40°$, find B.

From the formula, $$\frac{\sin B}{10}=\frac{\sin 40°}{12}.$$

$$\therefore \sin B=\frac{10 \sin 40°}{12}=\frac{6{\cdot}428}{12}=0{\cdot}5357.$$

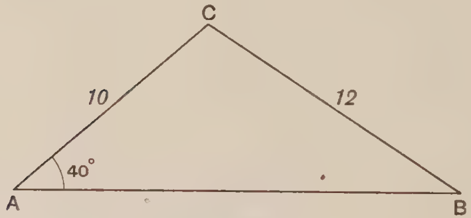


FIG. 190.

From the tables, $\sin\ 32°\,24'=0{\cdot}5357$;

$$\therefore \sin 147°\,36'=0{\cdot}5357;$$

$$\therefore B=32°\,24' \text{ or } 147°\,36'.$$

But $B=147°\,36'$ is impossible since

$$A=40° \quad \text{and} \quad A+B+C=180°;$$

$$\therefore B=32°\,24' \text{ is the only solution.}$$

Note. There is only one solution, because the given angle is opposite the larger of the two given sides.

EXERCISE IX. b.

Illustrate your answers for Nos. 1-12 by *rough* figures.

1. If $B=47°$, $C=53°$, $b=8{\cdot}61$, find c.
2. If $A=25°$, $B=79°\,30'$, $a=15{\cdot}6$, find b.
3. If $A=110°$, $B=29°$, $a=12{\cdot}4$, find b.
4. If $B=32°$, $C=99°\,20'$, $b=4{\cdot}28$, find c.
5. If $A=49°$, $B=77°$, $c=7{\cdot}46$, find a.

6. If $B=19°$, $C=43°$, $a=9{\cdot}36$, find c.

7. If $b=11{\cdot}2$, $c=8{\cdot}3$, $B=52°$, find C.

8. If $a=8{\cdot}45$, $b=6{\cdot}73$, $A=67°\ 45'$, find B.

9. If $a=15{\cdot}6$, $b=21{\cdot}7$, $B=112°$, find A.

10. If $a=9{\cdot}45$, $b=7{\cdot}32$, $A=121°\ 24'$, find B.

11. If $a=6{\cdot}31$, $c=8{\cdot}45$, $C=73°\ 15'$, find B.

12. If $a=9{\cdot}24$, $c=7{\cdot}48$, $A=37°\ 40'$, find C.

By drawing rough figures, find out whether two, one or no triangle can be drawn to fit the following data, Nos. 13-22.

13. $c=4$, $b=5$, $C=25°$.

14. $c=5$, $b=4$, $C=25°$.

15. $c=3$, $b=10$, $C=30°$.

16. $c=5$, $b=10$, $C=30°$.

17. $c=7$, $b=10$, $C=30°$.

18. $c=12$, $b=10$, $C=30°$.

19. $a=5$, $c=4$, $A=70°$.

20. $a=5$, $c=4$, $A=110°$.

21. $a=1$, $c=4$, $A=110°$.

22. $a=8$, $b=7$, $B=50°$.

23. $a=10$, $B=52°$; b is also given; what can you say about the value of b, if (i) two distinct triangles, (ii) only one triangle, (iii) no triangle can be drawn to fit the data?

24. $a=5$, $b=6$; in addition *either* A *or* B is given; in which case will the triangle be uniquely determined?

25. $b=10$, $C=115°$; in addition c is given; what can you say about c, if a solution is possible? Is more than one solution possible for any one given value of c?

Find all possible answers in Nos. 26-34; if there is no possible answer, say so.

26. If $a=6{\cdot}32$, $b=8{\cdot}47$, $A=43°$, find B.

27. If $b=12{\cdot}3$, $c=16{\cdot}9$, $B=51°$, find C.

28. If $a=3{\cdot}48$, $c=3{\cdot}37$, $C=68°$, find B.

29. If $a=7{\cdot}14$, $b=10{\cdot}3$, $A=57°$, find C.

30. If $b=5{\cdot}92$, $c=4{\cdot}73$, $C=53°\ 3'$, find B.

31. If $b=8{\cdot}46$, $c=7{\cdot}15$, $B=41°\ 24'$, find A.

32. If $b=7{\cdot}2$, $c=8{\cdot}1$, $B=127°$, find C.

33. If $b=3{\cdot}8$, $c=2{\cdot}9$, $B=117°\ 45'$, find A.

34. If $a=5{\cdot}61$, $c=4{\cdot}73$, $C=52°\ 27'$, find B.

Find the remaining sides and angles of the following triangles, Nos. 35-45.

35. $A=73^\circ\,20'$, $C=42^\circ\,50'$, $a=8{\cdot}23$.

36. $a=7{\cdot}81$, $b=6{\cdot}24$, $B=51^\circ\,15'$.

37. $A=29^\circ\,17'$, $B=32^\circ\,48'$, $c=3{\cdot}64$.

38. $b=8{\cdot}46$, $c=6{\cdot}38$, $B=127^\circ\,20'$.

39. $a=5{\cdot}18$, $b=6{\cdot}26$, $A=54^\circ\,35'$.

40. $B=32^\circ\,45'$, $C=111^\circ\,25'$, $a=4{\cdot}35$.

41. $a=7{\cdot}64$, $c=8{\cdot}23$, $C=63^\circ\,30'$.

42. $a=9{\cdot}92$, $b=7{\cdot}23$, $A=90^\circ$. **43.** $a=4{\cdot}87$, $c=9{\cdot}14$, $B=90^\circ$.

44. $a=513$, $c=724$, $C=132^\circ\,30'$. **45.** $b=804$, $c=640$, $C=39^\circ\,20'$.

The cosine formula.

In any triangle ABC,

$$\mathbf{c^2 = a^2 + b^2 - 2ab\cos C.}$$

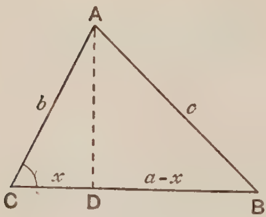


FIG. 191.

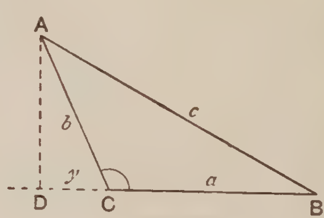


FIG. 192.

Draw AD perpendicular to BC.

In Fig. 191, $c^2 = BD^2 + DA^2 = (a-x)^2 + DA^2$

$$= a^2 - 2ax + x^2 + DA^2$$

$$= a2 - 2ax + b^2.$$

But $x = b\cos C$; $\therefore\ c^2 = a^2 + b^2 - 2ab\cos C$.

In Fig. 192, $c^2 = BD^2 + DA^2 = (a+y)^2 + DA^2$

$$= a^2 + 2ay + y^2 + DA^2$$

$$= a^2 + 2ay + b^2.$$

But $y = b\cos DCA = b\cos(180^\circ - C) = -b\cos C$;

$$\therefore\ c^2 = a^2 + b^2 - 2ab\cos C.$$

Note. It is worth while pointing out three special cases of this formula:

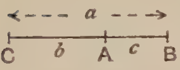


FIG. 193.

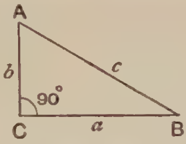


FIG. 194.

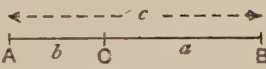


FIG. 195.

Consider two rods CA, CB jointed at C, with their ends A, B connected by an elastic string. Keep CB fixed and rotate CA.

Fig. 193 represents $\triangle$ACB when $C=0°$; $\cos 0°=1$, and the formula gives $c^2=a^2+b^2-2ab=(a-b)^2$; $\therefore\ c=a-b$.

In Fig. 194, $C=90°$; $\cos 90°=0$; the formula gives

$$c^2=a^2+b^2.$$

Fig. 195 represents $\triangle$ACB when $C=180°$; $\cos 180°=-1$, and the formula gives

$$c^2=a^2+b^2-2ab(-1)=(a+b)^2; \quad \therefore\ c=a+b.$$

Note. We have proved that the formula

$$c^2=a^2+b^2-2ab\cos C$$

is true in every triangle, whether C is acute or obtuse.

If $C=90°$, the formula is equivalent to Pythagoras' theorem.

If C is acute, cos C is positive; if C is obtuse, cos C is negative.

Consequently $c^2 < a^2+b^2$ if C is acute, and $c^2 > a^2+b^2$ if C is obtuse. In the same way, we can express a in terms of b, c, A and b in terms of c, a, B. We have, therefore, the following results, which must be committed to memory:

$$\mathbf{a^2=b^2+c^2-2bc\cos A} \quad \text{or} \quad \cos A=\frac{\mathbf{b^2+c^2-a^2}}{\mathbf{2bc}},$$

$$\mathbf{b^2=c^2+a^2-2ac\cos B} \quad \text{or} \quad \cos B=\frac{\mathbf{c^2+a^2-b^2}}{\mathbf{2ca}},$$

$$\mathbf{c^2=a^2+b^2-2ab\cos C} \quad \text{or} \quad \cos C=\frac{\mathbf{a^2+b^2-c^2}}{\mathbf{2ab}}.$$

By means of the cosine formula, we can solve a triangle, given *either* two sides and the included angle *or* three sides.

It often saves time to use a Table of Squares.

Example III. Given $b=4$, $c=5$, $\mathsf{A}=115°$, find a.

$$a^2=b^2+c^2-2bc\cos \mathsf{A}.$$

$\cos 115° = -\cos 65° = -0{\cdot}4226$;

$$\therefore\ a^2=4^2+5^2-2(4)(5)(-0{\cdot}4226)=16+25+40\times 0{\cdot}4226$$
$$=41+16{\cdot}90=57{\cdot}90;$$
$$\therefore\ a=\mathbf{7{\cdot}61}.$$

Example IV. Given $a=3{\cdot}82$, $c=5{\cdot}46$, $\mathsf{B}=37°\,25'$, solve $\triangle \mathsf{ABC}$.

$$b^2=c^2+a^2-2ca\cos \mathsf{B}=(5{\cdot}46)^2+(3{\cdot}82)^2$$
$$-2(5{\cdot}46)(3{\cdot}82)\cos 37°\,25'$$
$$=29{\cdot}81+14{\cdot}59-33{\cdot}13=44{\cdot}40-33{\cdot}13$$
$$=11{\cdot}27;$$

$$\begin{array}{|l}0{\cdot}3010\\0{\cdot}7372\\0{\cdot}5821\\\bar{1}{\cdot}8999\\\hline 1{\cdot}5202\end{array}$$

$$\therefore\ b=3{\cdot}357 \simeq \mathbf{3{\cdot}36}.$$

From the sine formula,

$$\frac{\sin \mathsf{A}}{3{\cdot}82}=\frac{\sin 37°\,25'}{3{\cdot}357};$$

$$\therefore\ \sin \mathsf{A}=\frac{3{\cdot}82\sin 37°\,25'}{3{\cdot}357};$$

$$\begin{array}{|l|l}0{\cdot}5821 & 0{\cdot}5229\\\bar{1}{\cdot}7837 & \\\hline 0{\cdot}3658 & \\0{\cdot}5259 & \\\hline \bar{1}{\cdot}8399 & \end{array}$$

$$\therefore\ \mathsf{A}=\mathbf{43°\,46'}.$$

[*Since* $a<c$, $\mathsf{A}<\mathsf{C}$; $\therefore$ A *cannot be obtuse*; *see Note* (ii) *below.*]

Lastly,

$$\mathsf{C}=180°-\mathsf{A}-\mathsf{B}=180°-43°\,46'-37°\,25'=180°-81°\,11';$$
$$\therefore\ \mathsf{C}=\mathbf{98°\,49'}.$$

Note. (i) If the data consist of *either* two sides and the included angle *or* three sides, it is necessary to use the cosine formula for the first operation; but *it is never necessary to use it twice.* Always continue with the sine formula, because it is quicker.

(ii) In the second operation, *always find the smaller of the two unknown angles*: this must be acute, and so there is no possibility of any ambiguity.

(iii) Four figures should be retained *throughout the working*, in order to secure as high a degree of accuracy in the answer as the tables permit.

Example V. Given $a=3{\cdot}46$, $b=5{\cdot}39$, $c=7{\cdot}12$, find C.

$$\cos C=\frac{a^2+b^2-c^2}{2ab}=\frac{(3{\cdot}46)^2+(5{\cdot}39)^2-(7{\cdot}12)^2}{2(3{\cdot}46)(5{\cdot}39)}$$

$$=\frac{11{\cdot}97+29{\cdot}05-50{\cdot}69}{6{\cdot}92\times5{\cdot}39}=\frac{41{\cdot}02-50{\cdot}69}{6{\cdot}92\times5{\cdot}39}$$

$$=-\frac{9{\cdot}67}{6{\cdot}92\times5{\cdot}39}.$$

0·9854	0·8401
1·5717	0·7316
$\bar{1}$·4137	1·5717

[Now if $\cos\theta=\dfrac{9{\cdot}67}{6{\cdot}92\times5{\cdot}39}$,

$\log\cos\theta=\bar{1}{\cdot}4137$; $\therefore\ \theta=74°\,59'$.]

$$\therefore\ C=180°-74°\,59'=105°\,1';$$

$$\therefore\ C\simeq\mathbf{105°}.$$

Note. (i) If you are asked to solve a triangle, given all three sides, *start by finding the smallest angle*; this avoids any difficulty arising from the cosine being negative. Then, as before, continue with the sine formula and *find next the smaller of the two remaining angles*; this avoids any possibility of ambiguity.

(ii) The portion in brackets is inserted to explain the argument; it would not appear in a formal solution.

(iii) If among the data there are either two equal sides or two equal angles, it is a waste of time to use either sine or cosine formula; the triangle is isosceles, and should be solved by drawing a perpendicular from the vertex to the base.

In early times, triangles were solved by inscribing them in circles and calculating the sides in terms of the radius ($a=2R\sin A$); the sine formula was known to Ptolemy, although not of course as expressed in modern notation. The cosine formula is equivalent to a theorem of Euclid, but its first explicit statement is due to Vieta (1593).

EXERCISE IX. c.

1. If $a=2$, $b=3$, $C=15°$, find c.
2. If $b=10$, $c=5$, $A=41° 27'$, find a.
3. If $b=2$, $c=5$, $A=124° 15'$, find a.
4. If $a=2$, $c=1$, $B=164° 18'$, find b.
5. If $a=4$, $b=3$, $c=2$, find B.
6. If $a=7$, $b=6$, $c=10$, find A.
7. If $a=5$, $b=6$, $c=9$, find C.
8. If $a=7$, $b=5$, $c=3$, find A.

Solve the following triangles, Nos. 9-26.

9. $a=2$, $b=5$, $C=21° 30'$.
10. $b=4$, $c=5$, $A=102° 8'$.
11. $a=6$, $c=10$, $B=15° 24'$.
12. $a=3$, $b=5$, $C=139° 33'$.
13. $a=6$, $b=5$, $c=3$.
14. $a=10$, $b=7$, $c=6$.
15. $a=100$, $b=80$, $c=50$.
16. $a=11$, $b=18$, $c=11$.
17. $a=8{\cdot}63$, $b=7{\cdot}42$, $C=37° 20'$.
18. $a=4{\cdot}17$, $b=5{\cdot}83$, $C=141° 25'$.
19. $a=114$, $b=137$, $c=184$.
20. $a=38{\cdot}2$, $b=21{\cdot}7$, $c=26{\cdot}3$.
21. $b=321$, $c=436$, $A=119° 15'$.
22. $a=8{\cdot}07$, $c=3{\cdot}14$, $B=22° 30'$.
23. $a=97$, $b=86$, $c=74$.
24. $a=4{\cdot}35$. $b=11{\cdot}91$, $c=9{\cdot}06$.
25. $a=73$, $b=89$, $c=73$.
26. $a=6{\cdot}8$, $c=6{\cdot}8$, $B=111° 30'$.

General procedure. Use the sine formula whenever possible. If you are given *either* 3 sides *or* 2 sides and the included angle, you must start with the cosine formula, but you should continue with the sine formula, using it to find the smaller of the two remaining angles. If you are given *either* 2 angles and one side *or* 2 sides and a not-included angle, the sine formula gives all that is required ; in the latter case a rough figure should be drawn as a guide to the nature of the solution.

Half-angle formulae. The half-angle formulae obtained in Chapter XVII. may be used for the solution of triangles, instead of the cosine formula. Geometrical proofs of these formulae are therefore given on p. 136A, etc., together with illustrative examples and an additional exercise.

MISCELLANEOUS EXAMPLES.

EXERCISE IX. d.

Solve the following triangles:

1. $A=97° 30'$, $B=42° 18'$, $c=123$. **2.** $a=59{\cdot}3$, $b=48{\cdot}6$, $c=37{\cdot}2$.

3. $a=112$, $b=84$, $B=47° 21'$. **4.** $a=6{\cdot}81$, $c=9{\cdot}06$, $B=119° 45'$.

5. $B=34° 16'$, $C=27° 33'$, $a=6{\cdot}35$. **6.** $a=183$, $b=102$, $c=124$.

7. $b=3{\cdot}81$, $c=5{\cdot}94$, $C=124° 15'$. **8.** $b=16{\cdot}9$, $c=24{\cdot}3$, $A=154° 18'$.

9. $a=18{\cdot}7$, $c=14{\cdot}2$, $C=37° 20'$.

10. $b=251$, $B=129° 15'$, $C=32° 50'$.

11. $a=1{\cdot}83$, $b=2{\cdot}49$, $c=3{\cdot}71$. **12.** $a=87{\cdot}2$, $A=127° 30'$, $C=32° 5'$.

13. $b=152$, $c=137$, $C=51° 30'$. **14.** $b=27{\cdot}4$, $c=36{\cdot}1$, $A=62° 35'$.

15. $a=36{\cdot}9$, $b=36{\cdot}9$, $A=59° 30'$.

Easy applications of the sine and cosine formulae.

EXERCISE IX. e.

1. London is 53° E. of N. from Winchester and 78° E. of S. from Oxford; Winchester is 51 miles due South of Oxford. Find the distances of Oxford and Winchester from London.

2. A base line AB, 1000 yards long, is measured on level ground running due South from A to B. The true bearings of a church C from A and B are 108° 10′ and 54° 30′ respectively. Find the distance of C from A.

3. A yacht starts from O and sails 8 miles due South and then 6 miles on a course 20° East of South. How far is she from O?

4. Two roads diverge from a point P at an angle of 28°. Two men leave P at the same time: one walks at 4 m.p.h. on one road, and the other bicycles at 12 m.p.h. along the other. How far apart are they (i) after 1 hour, (ii) after t hours?

5. A is a wireless station 35 miles due East of another station B. A ship in a fog discovers by wireless direction-finding that she is S. 20° E. of B and S. 50° W. of A. How far is she from B, to the nearest mile?

6. The elevation of the top of a tower is 29° from one point A and 48° from another point B, 100 feet nearer the foot of the tower which is in line with AB and at the same level. Find the height of the tower.

7. Three villages P, Q, R are connected by straight level roads; PQ = 5 miles, QR = 4 miles, $\angle$ PQR = 160°. How much is saved by going from P to R direct instead of *via* Q?

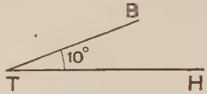


FIG. 196.

8. It is 400 yards from a tee T to a hole H, see Fig. 196; a golf ball driven from T lies at B, where TB = 200 yards, $\angle$ HTB = 10°. How far is the ball from the hole?

9. A "soccer" goal is 8 yd. wide; a man shoots when he is 18 yd. from one goal post and 20 yd. from the other. Within what angle must a ground-shot be made to score?

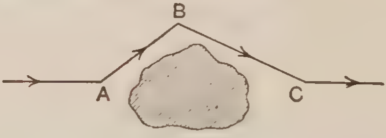


FIG. 197.

10. A scout moving due East turns at A (see Fig. 197) to avoid an obstacle and walks 120 yards to B on a bearing of 62° and then turns and walks on a bearing of 115° to C. What is the length of BC, if C is due East of A?

11. Mid-off stands 30 yd. from the batsman's wicket at an angle of 20° to the pitch. How far is he from the bowler's wicket?

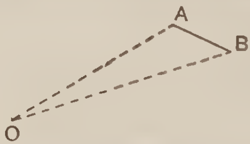


FIG. 198.

12. An officer at O (see Fig. 198) finds that the ends A, B of a belt of trees are 500 yd., 600 yd. away on magnetic bearings of 55°, 73° respectively. What is the length of AB, to the nearest 10 yd.?

13. A, B are two consecutive mile-stones on a straight road running East; from A, a church bears S. 61° E., and from B it bears S. 17° E. How many yards is it from A?

14. The front edge AB of a wood is 300 yd. long; a gun G is 2400 yd. from A and 2600 yd. from B. Through what angle must the gun be traversed to search the whole edge AB of the wood?

15. An officer at O (see Fig. 199) is observing for a battery B firing on a target at T; BO = 1500 yd., OT = 3000 yd.; the magnetic bearings of O from B and T from O are 70°, 56° respectively. Find the range BT.

16. Two ships leave harbour at noon in directions S. 62° W., S. 38° E. at 10, 12 knots respectively. How far apart are they at 12.45 p.m.?

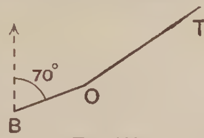


FIG. 199.

17. B, O, T represent the positions of a battery, an observer and the target respectively; ∠BOT = 102° 35′; BO = 1350 yd.; OT = 3113 yd. Find the range BT and the bearing of the line of fire if the bearing of B from O is 212° and if T is north of the line BO.

18. A, B, are two observers; A is 7000 yd. due west of B; A locates a battery on a bearing 52° 40′, and B locates it on a bearing 10°; how far is A from the battery?

19. In △ABC, $a = 3$, $b = 5$, $c = 7$; prove C = 120°.

20. Two searchlights A, B, $1\frac{1}{2}$ miles apart, are both directed on a Zeppelin C, vertically over the line AB; the elevations of the beams AC, BC are 76°, 46°. Find the height of the Zeppelin in feet.

21. A boat steaming due East is 3 miles away in a direction N. 30° E.; 5 minutes later, her direction is N. 50° E. What is her speed?

22. A road rises from A for 1 mile at an angle of 5° to the horizontal and then descends at an angle of 7° to the horizontal to the same level as A. How much longer to the nearest 10 yards is the uphill portion than the downhill portion?

23. A man AB, 6 ft. high, stands vertically on a hill-side (see Fig. 200), and his shadow BC falls on a slope of 25° when the sun's elevation is 57°. What is the length of BC?

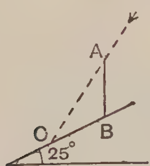


FIG. 200.

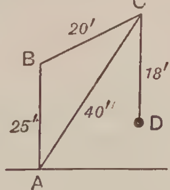


FIG. 201.

24. A crane ABC (see Fig. 201) carries a load D, as shown; AB is vertical. Find ∠BAC and the height of D above the level of A.

25. ABCD is a cyclic quadrilateral; AB = 5, BC = 4, CD = 7, DA = 6. Calculate ∠ABC

Harder applications of the sine and cosine formulae.

EXERCISE IX. f.

1. An observer O is 100 ft. away from the base A of a tower AB, and is on the same level as A; the tower has a spire BC; AB and BC subtend angles 42° and 12° at O. Find BC.

2. A road stretches from A 100 yards uphill at a slope of 5° to B; P is an object beyond B, and in the same vertical plane as AB; the elevations of P from A, B are 32°, 38°. Find the height of P above A.

3. A rectangular block (see Fig. 202) rests against an inclined plane OB; AOX is the ground line; OB = 2′. Find the heights of C, D above the ground.

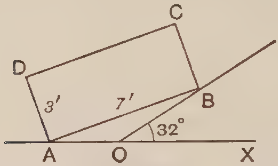


FIG. 202.

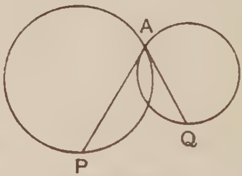


FIG. 203.

4. The radii of two circles (see Fig. 203) are 8, 6 inches, and their centres are 12 inches apart; AP, AQ are tangents. Calculate ∠ PAQ.

5. A window AB (see Fig. 204), pivoted at A, is held in position by a bar CD attached to it at D; small holes are punched in CD at intervals of 2 inches from D, and a peg P, fixed to the sill, passes through one of these holes. Find ∠ BAB′ through which the window is opened if (i) DP = 2″, (ii) DP = 4″.

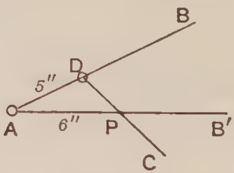


FIG. 204.

6. With the data of No. 5, find the least length of DC which enables the window to be fixed, when open at an angle of 80°.

7. In the triangle ABC, $a = 6$, $b = 8$, $c = 4$. Find the length of the line joining A to a point on BC 2 inches from C.

8. In a convex quadrilateral ABCD, AB = 5, BC = 3, CD = 3, DA = 4, AC = 6. Find the length of BD.

9. On a map, scale 1 inch to the mile, the distance between two villages A, B by road is shown as 6·74 inches; the road from A rises at a gradient of 1 in 10 for the first 2 miles and then descends at a steady gradient to B, at the same level as A. Find, to the nearest 100 yards, how much further the distance is from A to B by road than it appears to be on the map.

10. The crank OA (see Fig. 205) is free to turn about O, and the end P of the connecting rod AP is constrained to move along a line through O. Find the distances of P from its extreme positions when (i) $\angle AOP = 25°$; (ii) $\angle AOP = 155°$.

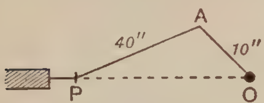


FIG. 205.

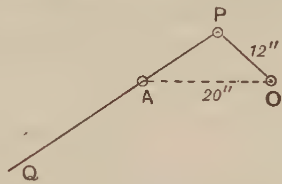


FIG. 206.

11. An arm OP (see Fig. 206) can rotate about O, and is hinged at P to a rod PQ, which can slide through a small fixed ring at A. Show that there are two positions of the mechanism for which $\angle OAP = 26°$. Find the distance between these two positions of P and the angle between the corresponding positions of OP.

12. Figure 207 represents the framework of a deck chair, whose shape is controlled by an adjustable arm BP. If OA, OC make angles of 30° with the ground, and if OB = 16″, BP = 20″, find the distance of P from O.

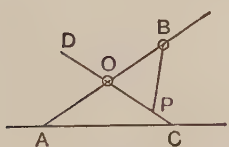


FIG. 207.

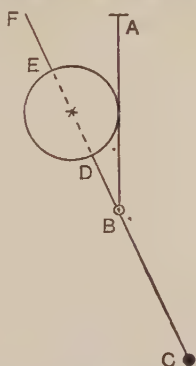


FIG. 208.

13. The mechanism in Fig. 208 consists of a fixed vertical bar AB, 30 inches long, with a bar FBC pivoted to AB at B and carrying a circular disc attached rigidly to it with its centre on FB; BC = 20 inches, BD = 10 inches, DE = 12 inches. Find the distance of A from C.

14. A triangular wedge ABC (see Fig. 209) is standing on an incline plane as shown : it would topple over if the median through A and the

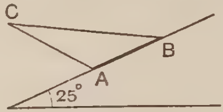


FIG. 209.

vertex C were on the same side of the vertical through A. AB = 10 inches ; ∠ABC = 38°. What is the greatest length of BC ?

15. In the framework in Fig. 210, AB = 6 ft. Calculate BD.

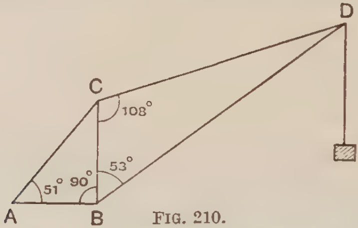


FIG. 210.

16. In the framework in Fig. 211, AB = 12 ft. Calculate BC.

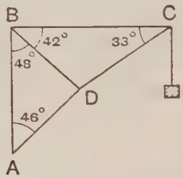


FIG. 211.

17. In the framework in Fig. 212, BE = 20 ft. Calculate AC.

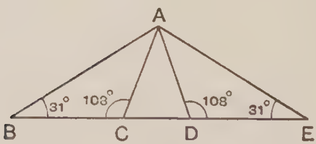


FIG. 212.

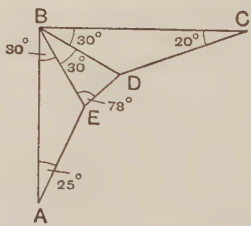


FIG. 213.

18. In the framework in Fig. 213, AB = 10 ft. Calculate BC.

19. A is North of B; P is East of A and bears N. 37° 30′ E. from B; Q is East of B and bears S. 72° 20′ E. from A. What is the bearing of P from Q?

20. In Fig. 214, prove that $BC = 2x \sin\theta$; then use the cosine formula, and so obtain $\cos 2\theta$ in terms of $\sin\theta$.

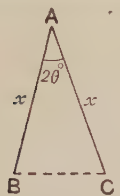


FIG. 214.

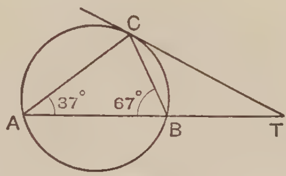


FIG. 215.

21. In Fig. 215, the tangent at C cuts AB at T. Calculate CT, given AB = 3·4 inches.

22. If, in Fig. 216, $\phi = 2\theta$, calculate the ratio $\dfrac{BC}{CD}$.

23. The sines of the angles of a triangle are in the ratio 5 : 6 : 7; prove that the cosines are in the ratio 25 : 19 : 7.

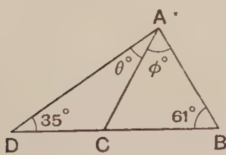


FIG. 216.

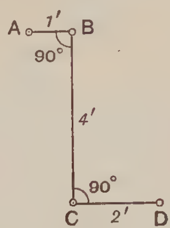


FIG. 217

24. The extremities A, D of the mechanism in Fig. 217 are fixed, and B, C are free joints. Calculate the angle ADC when AB is perpendicular to AD.

25. From a certain point two roads OA, OB run due West and West-Southwest. A boy-scout at O is ordered to go to a farm F 2 miles away down the western road, but by mistake goes along OB: after walking 2 miles he realises he is wrong, and taking a line across country arrives on the road OA after walking another mile. Should he now turn right or left to find F, and how much further must he walk?

26. The mechanism in Fig. 218 consists of three rods; AB and CD can turn about their ends A and D, which are fixed. Calculate the total angle through which AB can oscillate.

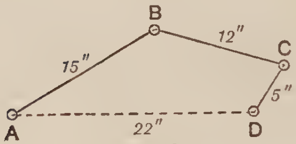


FIG. 218.

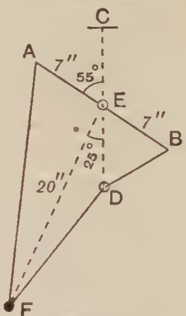


FIG. 219.

27. In Fig. 219, AB is a window which pivots about its centre E; CED is the window frame. The window is held open by two cords, one from B passing over a pulley at D and the other from A; the cords are attached to a peg F; FE = 20 inches and $\angle$FED = 25°. Find the length of each cord AF, BDF when the window is opened to an angle of 55°.

28. From an observation balloon at A, at an altitude of 10,000 feet, the angle of depression of a peak P (see Fig. 220) is 35°; the balloon sinks vertically 3000 feet to B, where the angle of depression of P is found to be 19°. What is the height of P above C?

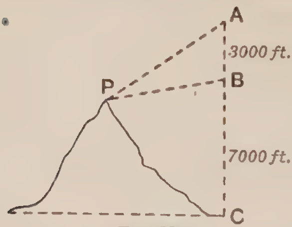


FIG. 220.

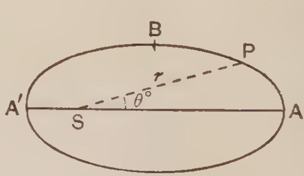


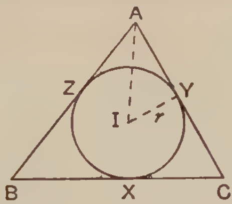
FIG. 221.

29. Fig. 221 represents an ellipse; S is a fixed point and ASA′ is a fixed line; if P is a variable point on the curve, the length of SP, r inches, is given by $r = \dfrac{12}{2 - \cos\theta^\circ}$, where $\angle$ASP $= \theta^\circ$; B is a point on the curve such that SB $= \frac{1}{2}$AA′. Calculate the lengths of SA, SA′, AB, A′B, and the angle ASB.

30. A flat triangular piece of wood has sides 5″, 6″, 7″. It is placed flat on a horizontal table and is then rotated through 30° about the 7″ side. What is the height of the opposite vertex above the table?

Alternative method.

I. *Given three sides of a triangle, to find its angles.*



From the results obtained on pp. 177-8, the area Δ of the triangle ABC is given by $\Delta = \sqrt{\{s(s-a)(s-b)(s-c)\}}$, where $s = \frac{1}{2}(a+b+c)$

Also, from p. 184, the radius r of the inscribed circle is given by $r = \dfrac{\Delta}{s}$. Now $\tan\dfrac{A}{2} = \tan \mathrm{IAY} = \dfrac{r}{\mathrm{AY}} = \dfrac{r}{s-a}$, see p. 186.

$$\therefore \tan\frac{A}{2} = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s(s-a)} = \sqrt{\frac{s(s-a)(s-b)(s-c)}{s^2(s-a)^2}};$$

$$\therefore \mathbf{\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}}.$$

In the same way, it may be proved that

$$\mathbf{\tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}}; \quad \mathbf{\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}}.$$

Example. Solve the triangle ABC, given that $a = 24{\cdot}76$, $b = 16{\cdot}38$, $c = 15{\cdot}12$.

		Logarithm.
$a = 24{\cdot}76$	$s - a = 3{\cdot}37$	0·5276
$b = 16{\cdot}38$	$s - b = 11{\cdot}75$	1·0701
$c = 15{\cdot}12$	$s - c = 13{\cdot}01$	1·1142
$2s = 56{\cdot}26$	$s = 28{\cdot}13$	1·4492
$s = 28{\cdot}13$		

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}};$$

$$\therefore \frac{A}{2} = 51^\circ\ 47';$$

$$\therefore \mathbf{A = 103^\circ\ 34'}.$$

1·0701	1·4492
1·1142	0·5276
2·1843	1·9768
1·9768	
2 | 0·2075	
0·1037	

$$\tan\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{s(s-b)}};$$

$$\therefore\ \frac{B}{2}=20^\circ\,0';$$

$$\therefore\ B=\mathbf{40^\circ\,0'}.$$

	1·1142	1·4492
	0·5276	1·0701
	1·6418	2·5193
	2·5193	
2	$\bar{1}$·1225	
	$\bar{1}$·5612	

$$\tan\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}};$$

$$\therefore\ \frac{C}{2}=18^\circ\,13';$$

$$\therefore\ C=\mathbf{36^\circ\,26'}.$$

	0·5276	1·4492
	1·0701	1·1142
	1·5977	2·5634
	2·5634	
2	$\bar{1}$·0343	
	$\bar{1}$·5171	

Check : $A+B+C=180^\circ$.

Note. (i) As previously explained, if 4-figure tables are used, the result will not necessarily be correct to the nearest minute.

(ii) It is useful to check the values of $s-a$, $s-b$, $s-c$ by adding them up, as above;

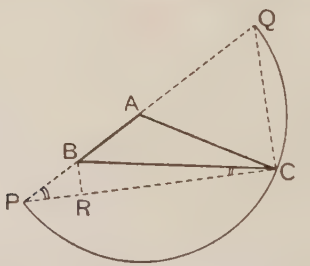
$$s-a+s-b+s-c=3s-(a+b+c)=3s-2s=s.$$

(iii) When A and B have been found, we can of course at once write down the value of C, since $C=180^\circ-(A+B)$.

(II.) *Given two sides and the included angle of a triangle, to find the remaining side and angles.*

To prove that
$$\frac{\tan\dfrac{B-C}{2}}{\tan\dfrac{B+C}{2}}=\frac{b-c}{b+c}.$$

Suppose AC > AB. With centre A and radius AC, describe



a circle, and let it cut AB produced at P, Q; join CP, CQ; draw BR perpendicular to CP.

Then $\quad PB = PA - BA = CA - BA = b - c.$

and $\quad BQ = BA + AQ = BA + AC = b + c.$

Also $\quad \angle BPR = \angle QPC = \frac{1}{2}\angle QAC = \frac{B+C}{2}$

and $\quad \angle BCR = \angle ABC - \angle BPC = B - \frac{B+C}{2} = \frac{B-C}{2};$

$$\therefore \frac{\tan\frac{B-C}{2}}{\tan\frac{B+C}{2}} = \frac{\tan BCR}{\tan BPR} = \frac{BR}{RC} \div \frac{BR}{RP} = \frac{PR}{RC}.$$

But $\angle QCP = 90°$, $\angle$ in semicircle; $\therefore$ BR is $\parallel$ QC;

$$\therefore \frac{PR}{RC} = \frac{PB}{BQ} = \frac{b-c}{b+c};$$

$$\therefore \frac{\tan\frac{B-C}{2}}{\tan\frac{B+C}{2}} = \frac{b-c}{b+c}.$$

In the same way, it may be proved that

$$\frac{\tan\frac{C-A}{2}}{\tan\frac{C+A}{2}} = \frac{c-a}{c+a}; \quad \frac{\tan\frac{A-B}{2}}{\tan\frac{A+B}{2}} = \frac{a-b}{a+b}.$$

Example. Solve the triangle ABC, given that

$$a = 24{\cdot}76,\ b = 16{\cdot}38,\ C = 36°\,26'.$$

$$\frac{\tan\frac{A-B}{2}}{\tan\frac{A+B}{2}} = \frac{a-b}{a+b}.$$

Now $\quad a - b = 24{\cdot}76 - 16{\cdot}38 = 8{\cdot}38,$

$a + b = 24{\cdot}76 + 16{\cdot}38 = 41{\cdot}14,$

$\frac{A+B}{2} = \frac{1}{2}(180° - C) = \frac{1}{2}(180° - 36°\,26')$

$= \frac{1}{2}$ of $143°\,34' = 71°\,47';$

$$\therefore \tan\frac{A-B}{2}=\frac{8\cdot38\tan 71^\circ 47'}{41\cdot14};$$

$$\therefore \frac{A-B}{2}=31^\circ 45';$$

$$\begin{array}{|l} 0\cdot9232 \\ 0\cdot4826 \\ \hline 1\cdot4058 \\ 1\cdot6142 \\ \hline \bar{1}\cdot7916 \end{array}$$

but $$\frac{A+B}{2}=71^\circ 47';$$

$\therefore$ adding, $A=\mathbf{103^\circ 32'}$, and subtracting, $B=\mathbf{40^\circ 2'}$.

Further, $$\frac{c}{\sin 36^\circ 26'}=\frac{16\cdot38}{\sin 40^\circ 2'};$$

$$\therefore c=\frac{16\cdot38\sin 36^\circ 26'}{\sin 40^\circ 2'}\simeq\mathbf{15\cdot1}.$$

$$\begin{array}{|l} 1\cdot2143 \\ \bar{1}\cdot7737 \\ \hline 0\cdot9880 \\ \bar{1}\cdot8084 \\ \hline 1\cdot179 \end{array}$$

The examples in Ex. IX. c. may now be solved by using the half-angle formulae. Further practice in their use is given in Ex. XVII. a. (p. 249); for the convenience of the reader, part of that exercise is reprinted below.

EXERCISE XVII. a.

Solve the following triangles, Nos. 1 16:

1. $a=63\cdot4$, $b=52\cdot7$, $c=78\cdot4$.
2. $a=1\cdot34$, $b=3\cdot47$, $c=2\cdot69$.
3. $a=5\cdot612$, $b=4\cdot381$, $c=7\cdot105$.
4. $a=11\cdot86$, $b=14\cdot13$, $c=19\cdot77$.
5. $a=8\cdot94$, $b=7\cdot32$, $C=52^\circ 38'$.
6. $b=31\cdot4$, $c=41\cdot5$, $A=72^\circ 44'$.
7. $a=6\cdot36$, $c=4\cdot78$, $B=124^\circ 26'$.
8. $a=11\cdot73$, $b=15\cdot64$, $C=104^\circ 48'$.
9. $b=7\cdot326$, $c=9\cdot814$, $A=49^\circ 40'$.
10. $a=5\cdot614$, $A=41^\circ 20'$, $B=59^\circ 17'$.
11. $a=5\cdot084$, $c=8\cdot613$, $B=59^\circ 45'$.
12. $a=14\cdot78$, $B=110^\circ 32'$, $C=47^\circ 10'$.
13. $a=127\cdot2$, $b=158\cdot5$, $c=193\cdot3$.
14. $a=17\cdot14$, $b=10\cdot65$, $A=112^\circ 17'$.
15. $a=1740$, $b=2125$, $c=1435$.
16. $a=208\cdot7$, $b=171\cdot8$, $A=42^\circ 31'$.

REVISION PAPERS. R. 19-26.

R. 19.

1. (i) What angle does the line joining the origin to the point (2, 5) make with the positive direction of the x-axis?

(ii) Repeat part (i) for the point $(-2, 5)$.

2. What can you say about θ, (i) if $\sin\theta°$ is positive and $\sec\theta$ is negative, (ii) if $\tan\theta°$ is greater than 1 and $\sin\theta$ is negative?

3. Find the value of $\dfrac{\tan 75° 28' \cos 14° 15'}{\sec 22° 41'}$.

4. The top of a sloping desk is a rectangle 40 in. by 24 in., and the 24 in. sides are inclined at 10° to the horizontal. Find the inclination of a diagonal to the horizontal.

5. In a triangle $a=10$ cm., $B=47°$, $C=73°$. Find b.

R. 20.

1. What is the angle between the line joining (1, 2) to (3, 5), and the line joining (1, 2) to (5, 6)?

2. The minute-hand of a clock, whose face is in a vertical plane, is 4 in. long. Find a formula for the distance of the tip from the central vertical line of the clock at t minutes past the hour. Evaluate the result when $t=10$, 20, 30, 40, 50, and interpret your answers.

3. If $\sin i=\mu \sin i'$, find i when $\mu=1{\cdot}12$ and $i'=47° 20'$.

4. In the jointed mechanism in Fig. 222, AP=AQ=3 in., PB=QC=6 in. and PO=OQ=2 in. Find the greatest value of

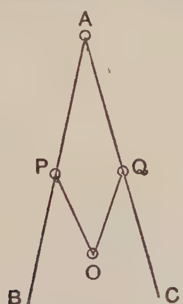


FIG. 222.

the angle BAC, and find the distance of O from the line BC when $\angle BAC=40°$.

5. Find the smallest angle of a triangle whose sides are 5 cm., 7 cm. and 8 cm.

R. 21.

1. Find the values of θ less than 360° if

(i) $\sin\theta° = 0{\cdot}432$; (ii) $\cos\theta° = 0{\cdot}417$; (iii) $\tan\theta° = 4$.

2. The ends of the link AB in Fig. 223 move on fixed lines OX, OY. Find the distance of P from these lines when $\angle OAB = 70°$.

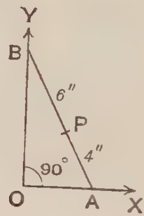


FIG. 223.

3. A stone thrown into the air with velocity u ft. per sec. at an angle of $a°$ to the horizontal will hit the ground at a distance $\dfrac{u^2 \sin 2a}{2g}$ ft., where $g = 32$.

If $u = 80$, draw a graph to show the distance reached for values of a from 0 to 90, and read from it (i) the greatest distance that can be reached, (ii) the values of a for which the distance is 56 ft.

4. Find the angles of a triangle whose sides are 9 cm., 9 cm., 10 cm.

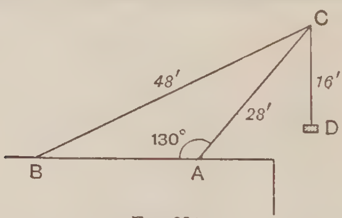


FIG. 224.

5. In the shear-legs shown in Fig. 224, find the height of the load D above the horizontal level of AB. Find also the length of AB.

R. 22.

1. Find the area of a triangle in which $b = 10{\cdot}4$ cm., $c = 8{\cdot}92$ cm., $A = 114° 22'$.

2. The hour-hand of a clock, whose face is in a vertical plane, is 5 in. long. Find a formula for the distance of the tip below its highest point after t hours. Evaluate the formula when t is 1, 3, 5, 7, 9 and 11.

3. A rectangular beam of wood is 10 ft. long, 18 in. wide and 12 in. deep. A cut is made across it at an angle of 74° to the top face, and at right angles to the side faces. Find the area of the surface thus exposed.

4. Fig. 225 is a diagram of the fairway of a dog-legged hole on a golf-course.

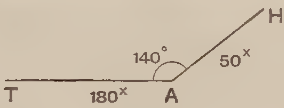


FIG. 225.

A golfer, who drives 220 yd., tries to drive direct from the tee T to the hole at H. How far short of H will his ball reach ?

5. With the data of Question 4, if another golfer who can only drive 200 yards drives from T, at what angle to TA should he drive so as just to reach the line of the fairway AH ?

R. 23.

1. The sine of an obtuse angle is $\frac{1}{3}$. Calculate its cosine without using tables.

2. Find the value of $\dfrac{\mathrm{P} \sin \alpha}{\sin (\alpha+\beta)}$, when $\mathrm{P}=10 \cdot 7$, $\alpha=52°$, $\beta=47°$.

3. A section of a shed is shown in Fig. 226. Find the inclination of the sloping roof to the horizontal.

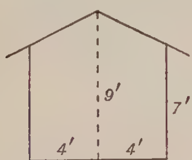


FIG. 226.

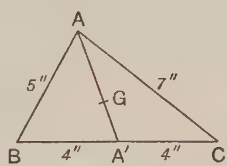


FIG. 227.

4. Two circles of radii 7 cm. and 8 cm. have their centres 10 cm. apart. Calculate the acute angle between the tangents at a point of intersection of the circles.

5. The centroid of a triangle is at a point G, such that AG = 2GA′, where A′ is the mid-pt. of BC. Find the length of AG in the triangle in Fig. 227.

R. 24.

1. In Fig. 228, AB is a diameter; TB, TP are tangents. Find the length of AP.

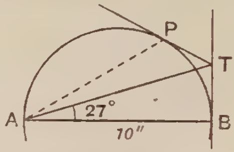


FIG. 228.

2. (i) What equation connects the acute angles θ and ϕ, if
$$\sin^2\theta + \sin^2\phi = 1 ?$$
(ii) Find a value of θ for which $\sin 3\theta^\circ = \cos 5\theta^\circ$.

3. A sphere of radius 4·32 inches rests in a conical funnel of vertical angle 68°, height 5·81 inches; the axis of the funnel is vertical and its rim is uppermost. How far does the top of the sphere project above the plane of the rim?

4. ABCD is a trapezium with AB and CD as parallel sides; AB = 3 in., BC = 4 in., CD = 8 in., $\angle$ BCD = 123°. Find the length of AD.

5. In $\triangle$ABC, AB = 5 in., AC = 4 in., $\angle$ BAC = 108°; the altitudes BE, CF of $\triangle$ABC intersect at H. Find the length of AH.

R. 25.

1. ABCD is a rhombus; AC = 7·3 in., $\angle$ABC = 162°. Find the length of BD.

2. Find a relation between x and y, independent of θ, given that $x = 1 + 2\tan\theta$, $y = 1 - 3\cot\theta$.

3. Two discs, centres A, B, rest in contact with each other, and a vertical wall OE, see Fig. 229; and the line AB makes an angle of 70°

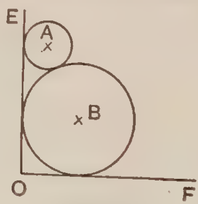


FIG. 229.

with the horizontal OF. B, whose radius is 10 cm., now rolls away from the wall and allows A to fall vertically. How far has B rolled when AB is inclined at 50° to the horizontal?

4. $\cos\theta° = -0{\cdot}67$. Find from the tables a value of $\sin 2\theta°$. Is there more than one possible value?

Find also from the tables the possible values of $\sin\left(\frac{\theta°}{2}\right)$.

5. In Fig. 230, find θ, if AC = BD.

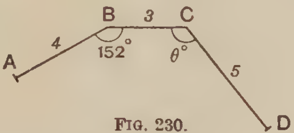


FIG. 230.

R. 26.

1. If $\sec a = \frac{m}{2} + \frac{1}{2m}$, prove that $\sec a + \tan a = m$ or $\frac{1}{m}$.

What is $\sec a - \tan a$?

2. The range R feet of a projectile up an inclined plane is given by the formula

$$R = \frac{V^2}{2g} \frac{\cos a \,.\, \sin (a - \beta)}{\cos^2 \beta}.$$

Calculate R when $V = 80$, $g = 32{\cdot}2$, $a = 48°$ and $\beta = 27°$.

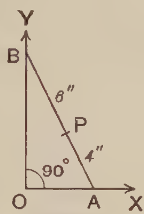


FIG. 223.

3. Find expressions for the coordinates of P (Fig. 223) referred to OX, OY as axes in terms of $\angle OAB = \theta$; and prove that as θ varies, these coordinates are connected by the equation $\frac{x^2}{36} + \frac{y^2}{16} = 1$.

4. In a triangle ABC, a, b and B are known, and there are two possible values c_1 and c_2 for the third side. Prove that

$$c_1 + c_2 = 2a \cos B.$$

5. A circular cam, radius 2″, rotates about a fixed axis C which is 1″ from O, the centre of the cam. As the cam rotates the bar AB

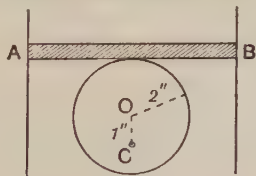


FIG. 231.

is raised and lowered, but remains horizontal. Find how far AB has descended when the cam has rotated through 100° from the position shown in the figure, where O is vertically above C.

CHAPTER X.

MENSURATION OF THE CIRCLE.

Circumference of circle.

All circles are of the same shape and are similar figures. Therefore the ratio $\frac{\text{circumference}}{\text{diameter}}$ is the same in all circles.

A rough idea of the value of this ratio can be obtained as follows:

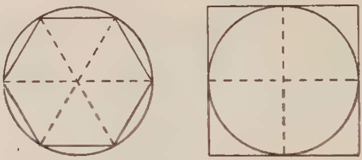


FIG. 232.

Take a circle, diameter d inches, and circumscribe a square about it, and inscribe a regular hexagon in it. Each side of the square is d in. long; $\therefore$ the perimeter of the square is $4d$ in.; each side of the hexagon is equal to the radius and is $\frac{d}{2}$ in. long; $\therefore$ the perimeter of the hexagon is $6 \times \frac{d}{2} = 3d$ in.

Hence $\frac{\text{circumference}}{\text{diameter}}$ is less than $\frac{4d}{d}$ and greater than $\frac{3d}{d}$, and therefore equals some number between 3 and 4.

We can find an approximate value of this number by experiment and measurement: and its value can be *calculated* to any required degree of accuracy; it is denoted by π; calculation gives

$$\pi = 3{\cdot}14159\ldots.$$

We therefore have $$\frac{\text{circumference}}{\text{diameter}} = \pi.$$

∴ the **circumference of a circle**, diameter d inches or radius r inches,

$$= \pi \mathbf{d} = 2\pi \mathbf{r} \textbf{ inches.}$$

Note. Archimedes proved that π lies between $3\frac{1}{7}$ and $3\frac{10}{71}$; in India it was often taken to be $\sqrt{10}$; in 1615 A.D., the value of π was calculated to 35 places of decimals by a German Professor, Ludolph van Ceulen; in 1853 A.D., William Shanks published its value to 707 places of decimals.

Area of circle. Let O be the centre of a circle of radius r inches.

Draw any polygon ABCDE... circumscribing the circle, and join O to the points of contact of AB, BC, CD, ... ; these joins are altitudes of the triangles OAB, OBC, OCD, ...

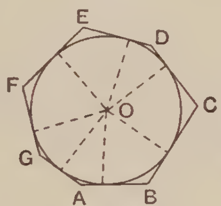


FIG. 233.

$$\text{Then area of polygon} = \triangle OAB + \triangle OBC + \triangle OCD + \ldots$$
$$= \tfrac{1}{2}r \cdot AB + \tfrac{1}{2}r \cdot BC + \tfrac{1}{2}r \cdot CD + \ldots$$
$$= \tfrac{1}{2}r(AB + BC + CD + \ldots)$$
$$= \tfrac{1}{2}r \times \text{perimeter.}$$

By increasing the number of sides of the polygon, it is possible to make the difference between the area of the polygon and the area of the circle as small as we please; and we say that in the limit,

$$\text{the } \textbf{area of the circle} = \tfrac{1}{2}r \times \text{perimeter of circle}$$
$$= \tfrac{1}{2}r \times 2\pi r = \pi \mathbf{r}^2 \textbf{ sq. inches.}$$

Note. A rigorous statement of the argument used above and rigorous definitions of what is meant by the length of a curved line or the area enclosed by a curved line are obviously unsuitable at this stage of the work.

Length of circular arc.

Equal arcs of a circle subtend equal angles at the centre O of the circle. Suppose, for example, arc CD = 3 arc AB; then

$\angle COD = 3\angle AOB$, because CD can be divided into three arcs each equal to AB. And in general

$$\frac{\text{arc PQ}}{\text{arc AB}} = \frac{\angle POQ}{\angle AOB}.$$

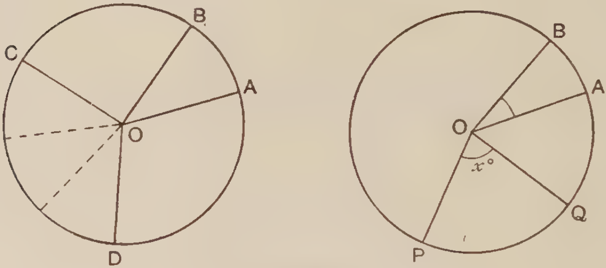


FIG. 234.

If the radius of the circle is r inches and $\angle POQ = x°$, then

$$\frac{\text{arc PQ}}{\text{circumference}} = \frac{x°}{360°};$$

$$\therefore \ \textbf{arc PQ} = \frac{x}{360} \times 2\pi r = \frac{\pi \mathbf{x}}{\mathbf{180}} \times \mathbf{r} \ \textbf{inches.}$$

Area of circular sector.

By the same argument,

$$\frac{\text{area of sector POQ}}{\text{area of sector AOB}} = \frac{\angle POQ}{\angle AOB};$$

$$\therefore \ \frac{\text{area of sector POQ}}{\text{area of circle}} = \frac{x°}{360°};$$

$$\therefore \ \textbf{area of sector POQ} = \frac{x}{360} \times \pi r^2 = \frac{\pi \mathbf{x}}{\mathbf{360}} \times \mathbf{r}^2 \ \textbf{sq. inches.}$$

Note that the area of sector $POQ = \frac{1}{2}r \times \frac{\pi x r}{180}$

$$= \tfrac{1}{2} \ \textbf{radius} \times \textbf{arc PQ.}$$

Curved surface of circular cylinder.

Suppose a circular cylinder (*e.g.* a round tin or a pencil with circular section) is of height h inches and radius r inches.

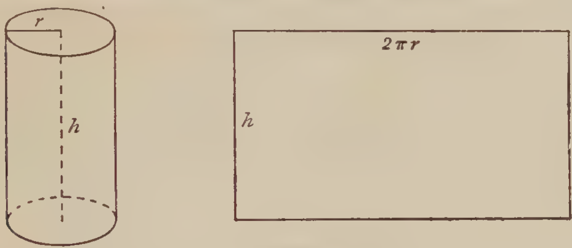


FIG. 235.

Take a sheet of paper the same height as the cylinder and wrap it round the curved surface and crease it so that the sheet just covers the cylinder without overlapping. When we unwrap it and fold it flat we obtain a rectangle of height h in. and breadth $2\pi r$ in.;

$$\therefore \text{ the area} = 2\pi rh \text{ sq. in.};$$

$\therefore$ **the area of the curved surface of the cylinder $= 2\pi rh$ sq. in.**

Volume of circular cylinder.

The volume of the cylinder = base-area × height

$$= \pi r^2 \times h = \pi r^2 h \text{ cu. in.}$$

Example I. Find the area of the minor segment cut off from a circle of radius 4 in. by a chord of length 6 in.

O is the centre of the circle; chord AB = 6 in.

Draw ON perp. to AB; let $\angle$ AON $= x°$.

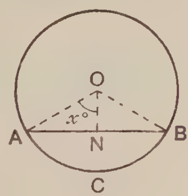


FIG. 236.

$$\text{Then } \sin x° = \frac{\text{AN}}{\text{AO}} = \tfrac{3}{4} = 0{\cdot}75; \quad \therefore x° = 48°\ 36';$$

$$\therefore \angle \text{AOB} = 2x° = 97°\ 12' = 97{\cdot}2°;$$

$$\therefore \text{area of sector AOB} = \frac{97{\cdot}2}{360} \times \pi \times 4^2$$

$$= 13{\cdot}57 \text{ sq. in.}$$

1·9877
0·4971
1·2041
3·6889
2·5563
1·1326

$$\text{Area of } \triangle AOB = \tfrac{1}{2}OA \,.\, OB \sin AOB = 8 \sin 97^\circ 12'$$
$$= 8 \sin 82^\circ 48' = 8 \times 0{\cdot}9921$$
$$= 7{\cdot}937 \,;$$

$$\therefore \text{ minor segment } ACB = 13{\cdot}57 - 7{\cdot}94 = 5{\cdot}63 \text{ sq. in.}$$
$$= \mathbf{5{\cdot}6} \text{ sq. in.}$$

Note. (i) Using 4-figure tables, we cannot, owing to the subtraction, rely on more than *two* significant figures in the answer.

(ii) It is sometimes convenient to use $\frac{22}{7}$ as a rough approximation for π. This value is correct to 3 figures, and results obtained from it are likely to be correct to 2 figures.

EXERCISE X. a.

1. A piece of fine cotton is wound 20 times round a cylinder and is then unwrapped and measured; its length is found to be 188·5 cm., the diameter of the cylinder is measured and found to be 3 cm. Find the value of $\dfrac{\text{circumference}}{\text{diameter}}$.

2. A circle of radius 8 cm. is drawn; steps of 1 cm. are taken round the circumference with a pair of dividers opened to 1 cm., and it is found that 50 such steps are required. Find the value of $\dfrac{\text{circumference}}{\text{diameter}}$ from this experiment.

3. A small wheel, radius 1·2″, is rolled along a straight line on a piece of paper and is found to travel a distance of 7·6″ in one revolution. Find the value of $\dfrac{\text{circumference}}{\text{diameter}}$ given by this experiment.

4. Fig. 237 represents two squares, one circumscribing the circle and the other inscribed in it. If the radius of the circle is r cm.,

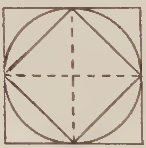


FIG. 237.

what are the areas of these squares? What does this tell you about the area of the circle?

5. Draw on squared paper two circles, one of radius 2 inches, the other of radius 3 inches. Find the area of each by counting the squares enclosed. What is the value of the ratio $\frac{\text{area of circle}}{\text{square of radius}}$ in each instance? If the answers disagree, which is likely to be the more accurate?

In the following questions π *may be taken either as* $\frac{22}{7}$ *or as* 3·142, *whichever is more convenient; log* π *may be taken to be* 0·4971. *Answers should never be given to more than* 3 *significant figures.*

6. Find the circumference and area of a circle (i) of radius 7 cm.; (ii) of diameter 4·7 cm.

7. Find the diameter of a circle (i) whose circumference is 11·34 in., (ii) whose area is 15·8 sq. in.

8. How many revolutions per mile are made by a wheel of diameter $3\frac{1}{2}$ ft.?

9. Find the speed of the earth in its orbit round the sun, in miles per sec., taking the orbit as a circle of radius 93,000,000 miles.

10. What is the area of the ring between two concentric circles of radii 7·3, 5·4 cm.?

11. The minute-hand of a church clock is 1 ft. 9 in. long. Find the distance its tip moves in 35 minutes.

12. An arc PQ of a circle of radius 8 cm. subtends 50° at the centre O. What is the length of the arc?

13. With the data of No. 12, find the area of the sector OPQ.

14. An arc of a circle of radius 7·3 cm. is 7·3 cm. long. What angle does the arc subtend at the centre?

15. A piece of flexible wire in the form of an arc of a circle of radius 4·2 in. subtends an angle of 30° at the centre of the circle; it is bent so as to form a complete circle. What is the radius of this circle?

16. The arcs in Fig. 238 are quadrants of circles. Prove that if the squares are equal, the shaded areas are equal.

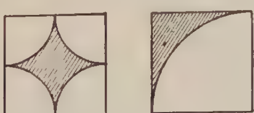


FIG. 238.

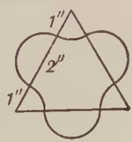


FIG. 239.

17. Fig. 239 shows an equilateral triangle, side 4″. Find the length of the curve, if it is composed of arcs of the same radius 1″.

18. A piece of wire 4 ft. long is bent into an arc of radius 1 ft. How far apart are the ends of the wire ?

19. A swing has ropes 14 ft. long, and when at rest the seat is 2 ft. above the ground ; the seat is prevented from rising more than 10 ft. above the ground. What is the length of the arc in which it can swing ?

20. What length of arc of a circle of radius 5 cm. is cut off by a chord of length 7 cm. ?

21. What is the area of the minor segment of a circle of radius 6 inches cut off by a chord of length 5 inches ? Also find this area from the *approximate* rule that

the area of a *small* segment $= \frac{2}{3}$ base $\times$ height.

22. What is the area of the major segment of a circle of radius 10 cm. cut off by a chord of length 12 cm. ?

23. A window consists of a rectangle surmounted by a semi-circle : its width is 5 ft. and its greatest height is 8 ft. Find (i) its area (ii) its perimeter.

24. Find (i) the volume, (ii) the *total* surface of a closed cylinder of height 6 in. and radius 4 in.

25. Find the diameter of a cylinder whose length is 10 feet and volume 300 cu. inches.

26. How many cylindrical glasses 3 in. in diameter can be filled to a depth of 4 in. from a cylindrical jug of diameter 6 in. and height 12 in. ?

27. A garden roller is 3 ft. in diameter and is 4 ft. wide. What area does it roll in 50 revolutions ?

28. A regular polygon of nine sides is inscribed in a circle of radius 10 cm. Calculate the difference between the area of the circle and the area of the polygon.

29. TA, TB are tangents to a circle of radius 4 in. ; $\angle ATB = 55°$.

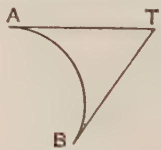


FIG. 240.

Calculate the area bounded by TA, TB and the arc AB. (Fig. 240.)

30. AB, AC, BC are arcs of circles of radii 4, 4, 5 inches, touching each other. Calculate (i) the area, (ii) the perimeter of Fig. 241.

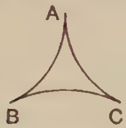


FIG. 241.

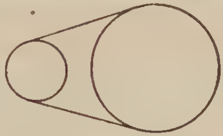


FIG. 242.

31. Two wheels of radii 1 ft., 3 ft., with their centres 5 ft. apart, are connected by a belt, see Fig. 242. Calculate the total length of the belt.

32. A screw thread is cut on the surface of a cylinder of diameter 4 cm.; the thread makes an angle of 72° with the axis of the cylinder. Find the length of thread if the cylinder is 50 cm. long; find also the number of turns it makes round the axis.

FIG. 243.

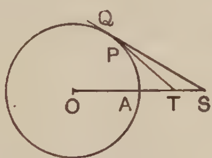


FIG. 244.

33. TP, SQ are tangents to the circle, centre O; OA = 4 cm., AT = TS = 2 cm. Calculate the length of the arc PQ. (Fig. 244.)

34. CAD is a tangent to the circle, centre O; $\angle$COA = 30°; CD = 3AO. Prove that 2BD is a close approximation for the length of the circumference of the circle. (Fig. 245.)

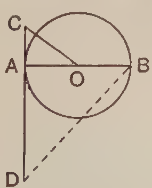


FIG. 245.

35. The two wheels of a cart are fixed 6 feet apart on an axle; the cart describes a circular course such that the diameter of the inner rut is 5000 ft. Find the difference between the lengths of the outer and inner ruts. Is any part of the data superfluous?

36. A circular disc, centre O, diameter 1 ft., is fixed flat on a table; a taut string AEFB joins two tacks A, B driven into the table;

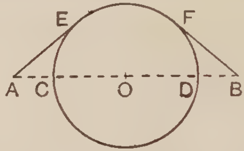


FIG. 246.

AC = DB = 3 in. and AOB is a straight line. Find the length of the string. (Fig. 246.)

37. The nut of a screw rises 3 inches in 10 turns; the diameter is 1 inch. Find the angle which the thread of the screw makes with the axis.

Latitude and longitude.

Let N, S represent the North and South Poles of the Earth and O its centre: the **Equator** is the section of the Earth's

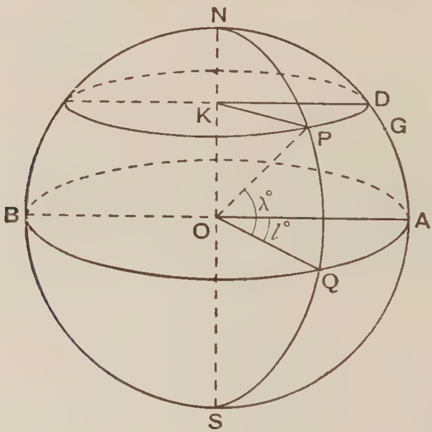


FIG. 247.

surface made by a plane through O perpendicular to NS. Any section of the Earth's surface by a plane through O is called a **Great Circle**. The great circles which pass through N and S

are called **Meridians**, and the particular meridian through Greenwich (G) is called the "Greenwich meridian." Let it cut the Equator at A, as shown.

Take any point P on the Earth's surface and draw the meridian through it, cutting the equator at Q.

Let $\angle AOQ = l^\circ$ and $\angle QOP = \lambda^\circ$.

Then, with the notation of the figure, **P is said to have latitude λ° North and longitude l° West.**

Latitudes vary from 90° S. (at the South Pole) to 0° (on the Equator) to 90° N. (at the North Pole).

Longitudes vary from 180° W. to 0° (on the Greenwich meridian) to 180° E.

Any section of the Earth's surface by a plane parallel to the Equator is called a **Parallel of Latitude**; all points on that small circle have equal latitudes.

In the figure K is the centre of the small circle PD, which is the parallel of latitude through P and cuts the Greenwich meridian at D.

It is important to notice that (i) $\angle PKD = \angle QOA = l^\circ$,
and that (ii) $\angle KPO = \angle POQ = \lambda^\circ$.

Hence, if the radius of the Earth $= a$ miles,

$$PK = a \cos \lambda^\circ \text{ miles,}$$

and $$\text{arc } PD = \frac{l}{360} \times 2\pi a \cos \lambda^\circ \text{ miles.}$$

A *nautical mile* is the length of an arc of the meridian which subtends an angle of $1'$ at the centre of the Earth.

Taking the radius of the Earth as 3960 statute miles, we see that

$$1 \text{ nautical mile} = \frac{1}{60 \times 360} \times 2\pi \times 3960$$
$$= \mathbf{1{\cdot}15} \text{ statute miles} = \mathbf{6080} \text{ feet.}$$

Two places on the same meridian whose latitudes differ by $1'$ are therefore 1 nautical mile or 1·15 statute miles apart.

But the distance between two places on the same parallel of latitude, measured along that parallel, whose longitudes differ by, say, $1'$ depends on their latitude, for the distance is an arc of a circle of radius $a \cos \lambda^\circ$; the greater λ is, the smaller this distance is.

Local time. The local time at any place P on the Earth's surface is 12 noon at the moment when the Sun appears to cross the meridian plane NPS of P. Thus if P is west of Greenwich, noon at P occurs after noon at Greenwich. If we know the correct local time at any given place and also know the corresponding Greenwich time, we can calculate the longitude of that place.

EXERCISE X. b.

[Take the radius of the Earth as 3960 statute miles.]

All distances are to be taken as measured along the earth's surface, unless otherwise stated.

1. Two places on the Equator are 150 nautical miles apart. What is the difference (i) in their longitudes, (ii) in their local times ?

2. Two places on the same meridian have latitudes (i) 23° N., 35° N. ; (ii) 10° N., 25° S. What is their distance apart (statute miles) ?

3. What is the length of the Equator in nautical miles ?

4. Reading and Greenwich have equal latitudes, 51° 28′ N., and the longitude of Reading is 59′ W. How far is Reading from Greenwich (statute miles) ?

5. What is the difference of local time between Paris (lat. 48° 50′ N., long. 2° 20′ E.) and Bombay (lat. 18° 55′ N., long 72° 54′ E.) ?

6. At the Equinox, when the Sun is vertical at the Equator, find the length of the shadow at mid-day in Winchester (lat. 51° 3′ N., long. 1° 18′ W.) of a vertical pole 10 feet high.

7. Eratosthenes found that the sun was in the zenith at Syene when it was 7° 12′ South of the zenith at Alexandria, which was known to be 5000 stadia North of Syene. What result did Eratosthenes deduce for the radius of the Earth (i) in stadia, (ii) in English miles, taking 1 stadium $= 606\frac{3}{4}$ ft. ?

8. In what latitude does a distance of 1 nautical mile measured along a parallel of latitude correspond to a difference of 3′ in longitude ?

9. A ship after sailing 200 (nautical) miles due West finds that her longitude has altered by 5°. What is her latitude ?

10. Find the distance travelled by the Eiffel Tower (lat. 48° 50′ N.) in 15 minutes, due to the Earth's rotation.

11. A, B are two points in latitude 52° N., whose longitudes differ by 20°. Find (i) the distance between A and B measured along the parallel of latitude, (ii) the angle which AB subtends at the centre of the Earth, (iii) the distance between A and B along a great circle.

Circular cone.

Let the base-radius of the cone be r in., the height h in., the slant side l in., and the semi-vertical angle $x°$.

Then $l^2=h^2+r^2$; $r=l \sin x°$; $h=l \cos x°$; $r=h \tan x°$;

and the perimeter of the base $=2\pi r$ in.

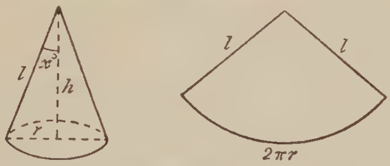


FIG. 248.

If we make a cut along a slant edge and unwrap the curved surface, we obtain a circular sector, radius l in., and bounded by an arc of length $2\pi r$ in.;

$\therefore$ area of sector $=\frac{1}{2}l \times 2\pi r=\pi r l$ sq. in.;

$\therefore$ **area of curved surface of cone** $=\pi$**rl** sq. in.

It can be proved by the methods of the calculus that the volume of any pyramid $=\frac{1}{3}$ base-area $\times$ height;

$\therefore$ **volume of cone** $=\frac{1}{3}\pi$**r²h** cu. in.

Frustum of a cone.

(i) **Volume.** Let the radii of the end-faces, centres E, F, of the frustum be a, b, and let the distance between the faces, be h. Figure

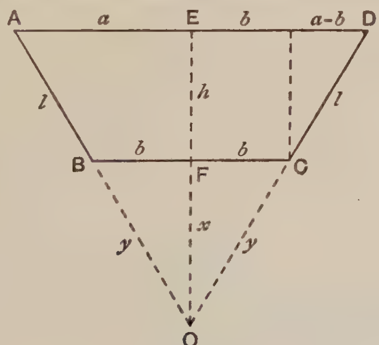


FIG. 249.

249 is a section through the axis; AB, DC meet at the vertex O of the cone from which the frustum is cut; let OF $=x$.

Volume of frustum $=\frac{1}{3}\pi a^2(h+x)-\frac{1}{3}\pi b^2 x$.

By similar triangles, $\dfrac{x}{b}=\dfrac{h}{a-b}$; $\therefore\ x=\dfrac{bh}{a-b}$;

$$\therefore\ x+h=\frac{bh}{a-b}+h=\frac{bh+ah-bh}{a-b}=\frac{ah}{a-b};$$

$$\therefore\ \text{volume of frustum}=\frac{\pi}{3}\left\{\frac{a^3h}{a-b}-\frac{b^3h}{a-b}\right\}=\frac{\pi h}{3}\times\frac{a^3-b^3}{a-b}$$

$$=\frac{\pi h}{3}(a^2+ab+b^2).$$

Note. If s_1, s_2 are the areas of the plane faces of the frustum, this formula for the volume may be written $\dfrac{h}{3}(s_1+\sqrt{s_1 s_2}+s_2)$. In this form, it is true for the frustum of any pyramid.

(ii) Area of curved surface. Let length of slant edge AB of frustum $=l$.

Let OB $=y$.

When unwrapped, the surface becomes a plane figure bounded by two concentric arcs of lengths $2\pi a$, $2\pi b$ and equal portions l of two radii.

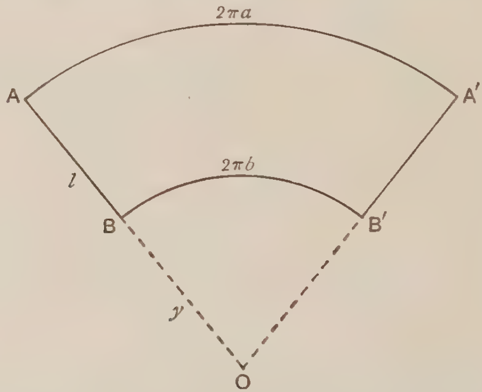


FIG. 250.

It is therefore equal to sector OAA′ − sector OBB′

$$=\pi a(l+y)-\pi b y.$$

By similar triangles (Fig. 249), $\dfrac{y}{b}=\dfrac{l}{a-b}$; $\therefore\ y=\dfrac{bl}{a-b}$.

Also, from Fig. 249, $$\frac{y+l}{y}=\frac{a}{b};$$

$$\therefore\ y+l=\frac{a}{b}y=\frac{al}{a-b};$$

$$\therefore\ \textbf{area of curved surface}=\frac{\pi a^2 l}{a-b}-\frac{\pi b^2 l}{a-b}=\pi l\left(\frac{a^2-b^2}{a-b}\right)$$
$$=\boldsymbol{\pi l\,.\,(a+b)}.$$

Note. This area can also be obtained by regarding the surface as the limit of the sum of a number of trapezia with the same height l. Thus, by applying the formula $\frac{1}{2}h(x+y)$ for the area of a trapezium (see p. 178), the result $\frac{l}{2}(2\pi a+2\pi b)$ can be obtained. This is the simplest way of remembering the result.

Sphere.

Suppose a sphere rests on the base of a cylindrical vessel which it just fits, *i.e.* the diameter of the sphere = internal diameter of cylinder.

Suppose any two planes are drawn parallel to the base; the surface of the sphere intercepted between these two planes is called

FIG. 251.

a **zone**, and Archimedes proved that the area of the zone is equal to the area of the surface intercepted between these two planes on the (inner) surface of the cylinder, circumscribing the sphere.

If the sphere is of radius r in., and if the distance between the parallel planes is d in., then

$$\text{area of zone of sphere}=2\pi rd \text{ sq. in.}$$

By taking $d=2r$, we obtain

$$\textbf{total area of surface of sphere}=2\pi r\times 2r=\boldsymbol{4\pi r^2} \text{ sq. in.}$$

We can regard the solid sphere as composed of a large number of pyramids, each with its vertex at the centre of the sphere and with a *small* portion of the surface of the sphere as base. We therefore can say that

$$\textbf{Volume of sphere}=\tfrac{1}{3}r\times\text{total area of surface of sphere}$$
$$=\tfrac{1}{3}r\times 4\pi r^2=\boldsymbol{\tfrac{4}{3}\pi r^3} \text{ cu. in.}$$

Example II. A sector of a circle of radius 5 in., angle of sector 110°, is bent into the form of a circular cone. Calculate the height, h in., and semi-vertical angle, $x°$, of the cone.

The arc of the sector $=\frac{110}{360}\times 2\pi\times 5$ in.

Let the radius of the base of the cone be r in.

Then $\qquad 2\pi r=\frac{110}{360}\times 2\pi\times 5;$

$\therefore\ r=\frac{55}{36}$ in.

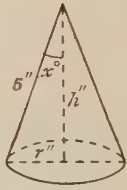


FIG. 252.

The slant edge of the cone $=5$ in.;

$$\therefore\ \sin x°=\frac{r}{5}=\frac{11}{36}=0\cdot3056;$$

$$\therefore\ x°=\mathbf{17°\ 48'};$$

$\therefore$ height of cone $=5\cos x°=5\cos 17°\ 48'=5\times 0\cdot9521$
$=\mathbf{4\cdot76}$ in.

EXERCISE X. c.

1. Find (i) the volume, (ii) the area of the curved surface of a cone, height 8 cm., base-diameter 5 cm.

2. The base of a conical tent is 14 ft. in diameter and its height is 8 ft. Find (i) the volume of the tent, (ii) the area of canvas required for making it.

3. Find the volume of a cone, vertical angle 54°, base-diameter 4 inches.

4. The curved surface of a cone of height 5″, base-radius 6″, is folded out flat. What is the angle of the sector so formed ?

5. A sector of angle 80° is bent into the form of a circular cone. Find the vertical angle of the cone.

6. Find (i) the volume, (ii) the area of the surface of a sphere of diameter 3 inches.

7. Taking the radius of the Earth as 3960 miles, find (i) the area of the Earth's surface, (ii) the area between the parallels of latitude, 30° N. and 59° N.

8. Find (in inches) the diameter of a sphere of volume 1 cu. ft.

9. Find (in inches) the diameter of a sphere whose surface area is 1 sq. ft.

10. A cylinder of diameter d inches contains water. Three spheres each of diameter d inches are placed in the cylinder. If all are submerged and no water overflows, find the height the water-level rises.

11. Find the area of the Earth's surface within the Arctic Circle, *i.e.* in latitudes North of 66° 32′ N.

12. A cylindrical boiler has hemi-spherical ends; its diameter is 4 ft. and its total internal length is 12 ft. Find its volume and its internal surface.

13. Spherical balls, each of diameter $1\frac{1}{2}$ in., are packed in a box measuring 6 in. by 3 in. by 3 in.; how much free space is there in the box, if as many are packed as possible?

14. Fig. 253 represents a hemi-spherical bowl of radius 8 inches, containing water to a depth of 3 inches. Find

(i) $\angle$ POQ;
(ii) the area of the wetted surface PCQ;
(iii) the volume of the sector of the sphere whose base is the wetted surface;
(iv) the volume of the cone, vertex O, base PQ;
(v) the volume of the water in the bowl.

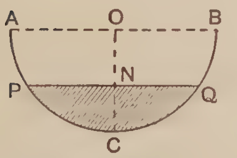


FIG. 253.

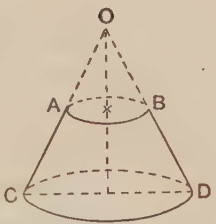


FIG. 254.

15. Fig. 254 represents a frustum of a cone whose faces have diameters 5 cm., 8 cm., and are 6 cm. apart. Find (i) the volume of the frustum, (ii) the vertical angle of the cone of which it forms part, (iii) the area of the curved surface of the frustum.

16. A solid consists of a cone mounted on a hemi-spherical base. Find the vertical angle of the cone if the volumes of the conical and spherical portions are equal. Fig. 255.

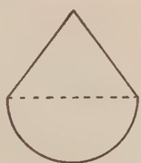


FIG. 255.

17. A chemist's measuring glass is conical in shape; it is 8 cm. deep and 3 cm. across the mouth. Calculate the distance on the slant edge between the markings for 1 c.c. and 2 c.c.

18. Find the radius of the sphere inscribed in a circular cone whose slant side is y cm. and semi-vertical angle $2\theta°$.

19. A sector of a circle of angle $\theta°$ is bent into the form of a cone of semi-vertical angle $\phi°$; obtain a relation between θ and ϕ.

20. A, B are diametrically opposite points on the base of a cone of semi-vertical angle 35° and slant edge 6 inches. Find the difference between the distances of B from A, (i) measured round the rim of the base, (ii) measured along the shortest path across the curved surface of the cone. [Unwrap the curved surface, as on p. 153.]

21. The section of a tube railway is the major segment of a circle of radius 6 ft. cut off by a chord of length 8 ft. What is the area of the inner curved surface of a tunnel 400 yd. long ?

22. A tumbler is a frustum of a cone 4 in. deep; the diameters of its upper and lower ends are 3 in., 2 in., and it contains water to a depth of 1 in. How many spherical shot of diameter $\frac{1}{10}$ inch must be poured in to raise the level half an inch ?

23. A solid sphere of diameter 10 cm. is melted down and recast as a hollow sphere whose thickness is $\frac{1}{4}$ of its outside radius. Find the area of its outer surface.

24. The thickness of a metal spherical shell is t in., and its mean radius is r in.; prove that the volume of the metal is $\pi t\left(4r^2+\frac{t^2}{3}\right)$.

25. Find the parallel of latitude which divides the surface of the Earth in the ratio 4 : 1.

26. In Fig. 256, PCQ represents a segment of height CN $=d$ in. of a sphere of radius a in.; O is the centre of the sphere. Prove that (i) the volume of the cone OPQ is $\frac{1}{3}\pi d(a-d)(2a-d)$ cu. in., (ii) the volume of the spherical sector OPQ is $\frac{2\pi a^2 d}{3}$ cu. in., (iii) the volume of the spherical segment PCQ is $\frac{1}{3}\pi d^2(3a-d)$ cu. in.

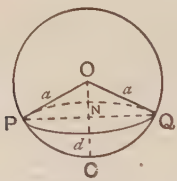


FIG. 256.

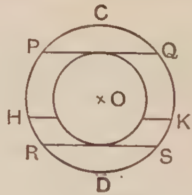


FIG. 257.

27. Fig. 257 represents two concentric spheres of radii a and $a-d$ inches. PQ and RS are parallel tangent planes. A variable plane HK parallel to PQ and RS is drawn between them; prove that the area intercepted on HK *between the spheres* is constant. Hence show that the space between the spheres and PQ and RS is

$$2\pi d(a-d)(2a-d).$$

Hence prove that the volume of the spherical segment PCQ is $\frac{1}{3}\pi d^2(3a-d)$ cu. in., as in No. 26.

CHAPTER XI.

CIRCULAR MEASURE.

Ratios of small angles. AB is a diameter of a circle, centre **O**, radius r in.; AP is an arc subtending $x°$ at O; the tangent at **P** meets BA produced at **T**.

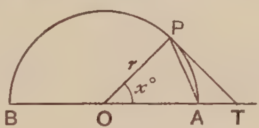


FIG. 258.

$$\text{Area of } \triangle AOP = \tfrac{1}{2}OA \,.\, OP \sin AOP = \tfrac{1}{2}r^2 \sin x°.$$

$$\text{Area of sector } AOP = \frac{x}{360} \times \pi r^2 = \frac{\pi r^2 x}{360}.$$

$$\text{Area of } \triangle TOP = \tfrac{1}{2}TP \,.\, OP = \tfrac{1}{2}r \tan x° \,.\, r = \tfrac{1}{2}r^2 \tan x°.$$

But $\triangle AOP <$ sector $AOP < \triangle TOP$;

$$\therefore\ \tfrac{1}{2}r^2 \sin x° < \frac{\pi r^2 x}{360} < \tfrac{1}{2}r^2 \tan x° \text{ or } \tfrac{1}{2}r^2 \frac{\sin x°}{\cos x°}.$$

Divide by $\frac{1}{2}r^2 \sin x°$; $\therefore\ 1 < \dfrac{\pi x}{180 \sin x°} < \dfrac{1}{\cos x°}$.

Now we can make $\cos x°$ take a value as near 1 as we like by making $x°$ a sufficiently small angle, for $\cos 0° = 1$.

$\therefore\ \dfrac{\pi}{180} \times \dfrac{x}{\sin x°}$ lies between 1 and a number which is greater than 1, but can be made as near 1 as we please by reducing sufficiently the size of the angle $x°$.

$$\therefore \text{ if } x° \text{ is a small angle, } \frac{\pi}{180} \times \frac{x}{\sin x°} \simeq 1;$$

$$\therefore\ \sin x° \simeq \frac{\pi}{180} \times x,$$

and, the smaller x is, the less is the percentage error of this approximation.

The Tables may be used to illustrate this result:

From the Tables, $\sin 5° = 0{\cdot}0872$; also $\frac{\pi}{180} \times 5 = \frac{\pi}{36} = 0{\cdot}0873$.

The awkward numerical factor, $\frac{\pi}{180}$, which occurs in this formula, also appeared in some of the formulae of the last chapter;

e.g. the length of an arc (p. 144) $= \frac{\pi}{180} \times rx$,

and the area of a circular sector $= \frac{\pi}{180} \times \frac{1}{2}r^2x$.

There are many other formulae of the same kind in which this factor appears; it is due to the unit (degrees) in which the angle $x°$ is measured. *We can simplify these formulae, making them easier to remember and easier to work with, by choosing a new unit for measuring angles.*

A radian.

O is the centre of a circle of radius r inches; AP is an *arc* of length equal to the radius, *i.e.* arc AP $= r$ inches.

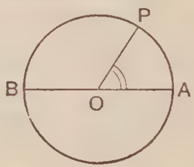


FIG. 259.

Then $\angle$POA is said to equal 1 radian (often written 1^c).

Let AB be a diameter; suppose $\angle AOP = x°$;

Then
$$\frac{x}{180} = \frac{\text{arc AP}}{\text{semicircle APB}} = \frac{r}{\pi r} = \frac{1}{\pi};$$

$$\therefore\ x = \frac{180}{\pi};$$

$$\therefore\ 1 \text{ radian} = \frac{180}{\pi} \text{ degrees} \simeq 57°\ 17{\cdot}7',$$

and π radians $= 180$ degrees.

Note. The size of a radian does not depend on the radius r of the circle, used in the definition; if it did so, it would be little use as a unit for angle-measure

The system of measuring angles in radians is called "Circular Measure," and the number of radians in an angle is often called the "circular measure" of the angle.

Radians and degrees. The fundamental relation

$$\pi \textbf{ radians} = \textbf{180 degrees}$$

enables any angle expressed in either unit to be converted to the other. We have at once

$$\theta \text{ radians} = \frac{180\theta}{\pi} \text{ degrees and } x \text{ degrees} = \frac{\pi x}{180} \text{ radians.}$$

Tables have, however, been constructed to save the time which this arithmetical calculation requires. (See end of book.)

The following special relations should be noted:

$$360° = 2\pi^c; \; 90° = \frac{\pi^c}{2}; \; 45° = \frac{\pi^c}{4}; \; 120° = \frac{2\pi^c}{3}; \; 60° = \frac{\pi^c}{3}; \; 30° = \frac{\pi^c}{6}.$$

The reader should make himself familiar with these results.

It is customary to speak of an "angle π" or an "angle $\frac{\pi}{2}$," etc., as short for π radians (π^c) or $\frac{\pi}{2}$ radians, etc. *When the unit is not named explicitly, it is usually implied that the angle is measured in radians.*

We shall now show how the formulae mentioned above are simplified by working in radians instead of in degrees.

Length of arc: area of sector.

Let O be the centre of a circle of radius r inches; let PQ be an arc subtending θ radians (θ^c) at O.

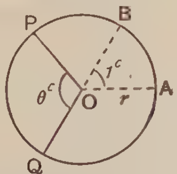


FIG. 260.

Let $\angle$AOB = 1 radian; $\therefore$ arc AB = r in.

But $$\frac{\text{arc PQ}}{\text{arc AB}} = \frac{\theta^c}{1^c}; \quad \therefore \frac{\text{arc PQ}}{r} = \theta.$$

$$\therefore \text{ arc PQ} = \mathbf{r\theta} \textbf{ inches.}$$

Again, since 2π radians $= 4$ right angles,

$$\frac{\text{sector POQ}}{\text{area of circle}} = \frac{\theta^c}{2\pi^c}; \quad \therefore \text{ sector POQ} = \pi r^2 \times \frac{\theta}{2\pi} \text{ sq. in.};$$

$$\therefore \text{ sector POQ} = \tfrac{1}{2}\mathbf{r^2\theta} \textbf{ sq. in.}$$

Ratios of small angles.

Use the same figure as on p. 159, taking $\angle \text{AOP} = \theta^c$.

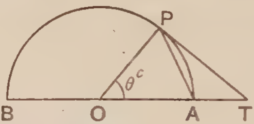


FIG. 261.

Since area of sector AOP $= \frac{1}{2}r^2\theta$, we have as before

$$\tfrac{1}{2}r^2 \sin\theta^c < \tfrac{1}{2}r^2\theta < \tfrac{1}{2}r^2 \frac{\sin\theta^c}{\cos\theta^c};$$

$$\therefore 1 < \frac{\theta}{\sin\theta^c} < \frac{1}{\cos\theta^c};$$

$\therefore$ as before, if θ^c is a small angle,

$$\mathbf{\sin\theta^c \simeq \theta.}$$

The Tables may be used to illustrate this approximation. If $\theta = \frac{1}{5}$, $\sin \frac{1}{5}^c \simeq \sin 11^\circ 28' \simeq 0{\cdot}199$; while $\frac{1}{5} = 0{\cdot}2$.

Further, if θ^c is a small angle, $\cos\theta^c \simeq 1$, and so

$$\tan\theta^c = \frac{\sin\theta^c}{\cos\theta^c} \simeq \sin\theta^c \simeq \theta;$$

$$\therefore \mathbf{\tan\theta^c \simeq \theta}, \text{ if } \theta^c \text{ is a small angle.}$$

From the Tables, if $\theta = \frac{1}{5}$,

$$\tan \tfrac{1}{5}^c \simeq \tan 11^\circ 28' \simeq 0{\cdot}203; \text{ while } \tfrac{1}{5} = 0{\cdot}2.$$

Note. (i) These two numerical examples illustrate the facts proved above that $\sin\theta^c < \theta$ and $\tan\theta^c > \theta$.

(ii) On p. 160 we took $x=5$ as an example of a small angle because it corresponded to 5°; but the angle corresponding to $\theta=5$ would be 5 radians $\simeq 286°$, which is not small.

Example I. Express $0{\cdot}35^c$ in degrees and 37° 20′ in radians.

(i) $$0{\cdot}35^c = \frac{0{\cdot}35 \times 180}{\pi} \text{ degrees} = 20{\cdot}06°$$

$$= \mathbf{20° \, 4'}.$$

(ii) $$37° \, 20' = 37{\cdot}33° = \frac{37{\cdot}33 \times \pi}{180} \text{ radians}$$

$$= \mathbf{0{\cdot}651^c}.$$

$\bar{1}{\cdot}5441$
$2{\cdot}2553$
$\bar{1}{\cdot}7994$
$0{\cdot}4971$
$1{\cdot}3023$

$1{\cdot}5720$
$0{\cdot}4971$
$2{\cdot}0691$
$2{\cdot}2553$
$\bar{1}{\cdot}8138$

Note. These results can be obtained direct from the printed conversion tables.

Example II. Find the length of the chord PQ which cuts off an arc 12 cm. long from a circle, centre O, radius 5 cm.

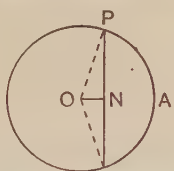


FIG. 262.

arc PAQ $= 12$ cm.; $\therefore \angle POQ = \frac{12}{5} = 2{\cdot}4$ radians.

From the Tables, $1^c = 57° \, 18'$, $1{\cdot}4^c = 80° \, 13'$;

$\therefore \angle POQ = 137° \, 31'$.

Draw ON perpendicular to PQ; then $\angle PON = \frac{1}{2}\angle POQ = 68° 45'$;

$\therefore$ PQ $=$ 2PN $= 2 \times 5 \sin 68° \, 45' = 10 \times 0{\cdot}9320$

$= \mathbf{9{\cdot}32}$ cm.

EXERCISE XI. a.

1. Calculate the following angles, in degrees and minutes:

(i) 1^c; (ii) 3^c; (iii) $\frac{1}{2}^c$; (iv) $1{\cdot}1^c$;
(v) $2{\cdot}5^c$; (vi) $0{\cdot}8^c$; (vii) $0{\cdot}07^c$; (viii) $0{\cdot}004^c$.

Use the Tables to check your answers.

2. Express in degrees the angles whose circular measures are:

(i) $\frac{\pi}{2}$; (ii) $\frac{3\pi}{4}$; (iii) $\frac{\pi}{5}$; (iv) $\frac{5\pi}{6}$; (v) $\frac{\pi}{12}$;
(vi) $\frac{3\pi}{8}$; (vii) $\frac{3\pi}{2}$; (viii) $\frac{7\pi}{4}$; (ix) $\frac{4\pi}{9}$; (x) $\frac{7\pi}{12}$.

3. Express the following angles in radians in terms of π:
(i) 270° ; (ii) 60° ; (iii) 150° ; (iv) 135° ; (v) 75° ;
(vi) 36° ; (vii) 108° ; (viii) 22° 30′; (ix) 315° ; (x) 210°.

4. Express in radians the following angles, and compare your answers with the values given in the Tables:
(i) 17° ; (ii) 39° ; (iii) 58° ; (iv) 86° ;
(v) 35′ ; (vi) 54′ ; (vii) 17° 54′ ; (viii) 46° 36′ ;
(ix) 74° 25′ ; (x) 127° 44′.

5. In some Tables we find $1^c = 57{\cdot}30°$, $0{\cdot}1^c = 5{\cdot}73°$, $0{\cdot}01^c = 0{\cdot}57°$, $0{\cdot}001^c = 0{\cdot}06°$. Use these results to express in degrees to one place of decimals (i) $1{\cdot}323^c$, (ii) $0{\cdot}435^c$; (iii) $2{\cdot}213^c$.

6. The radius of a circle is 10 cm. Find the length of an arc which subtends at the centre an angle of (i) 2^c, (ii) $1{\cdot}34^c$, (iii) $0{\cdot}55^c$, (iv) 37° 35′ (use Tables), (v) 157° 24′ (use Tables).

7. The radius of a circle is 4 inches. Find in radians the angle subtended at the centre by an arc of length (i) 3 in., (ii) 5 in., (iii) 1 ft., (iv) 2·36 in.

8. Use Tables to express 37° in radians, and write down the area of a sector of a circle, radius 10 cm., angle of sector 37°.

9. Write down the values of the following, the unit being a radian:

(i) $\sin\frac{\pi}{2}$; (ii) $\cos\pi$; (iii) $\tan\frac{3\pi}{4}$; (iv) $\sin\frac{3\pi}{2}$:
(v) $\cos\frac{\pi}{3}$; (vi) $\sin\frac{5\pi}{6}$; (vii) $\cos 2\pi$; (viii) $\sin\frac{2\pi}{3}$;
(ix) $\tan\frac{4\pi}{3}$; (x) $\cot\frac{5\pi}{4}$; (xi) $\cos\frac{4\pi}{3}$; (xii) $\sin\frac{3\pi}{4}$.

10. Simplify the following, the unit being a radian:

(i) $\sin(\pi-\theta)$; (ii) $\cos\left(\frac{\pi}{2}-\theta\right)$; (iii) $\tan\left(\frac{\pi}{2}+\theta\right)$;
(iv) $\cos(\pi+\theta)$; (v) $\sin(2\pi-\theta)$; (vi) $\tan(\pi-\theta)$;
(vii) $\sin\left(\frac{3\pi}{2}-\theta\right)$; (viii) $\cos\left(\frac{3\pi}{2}+\theta\right)$; (ix) $\cot(2\pi-\theta)$;
(x) $\tan(\pi+\theta)$; (xi) $\cos\left(\frac{\pi}{2}+\theta\right)$; (xii) $\sin\left(\frac{3\pi}{2}+\theta\right)$.

11. AB is an arc 9 in. long in a circle of radius 6 in., centre O. What angle is subtended by the chord AB, (i) at O, (ii) at a point on the major arc AB, (iii) at a point on the minor arc AB? Give the answers in radians.

12. The arc PQ of a circle, centre O, radius 5 in., is 6 in. long. Express $\angle POQ$ in radians; use Tables to convert this to degrees; then calculate the length of the chord PQ.

13. The area of the sector POQ of a circle, centre O, radius 10 cm., is 60 sq. cm. Express $\angle POQ$ in radians. Calculate the length of the chord PQ.

14. A wheel of radius 20 inches is spinning on its axis at 3 radians per second. Find the speed of a point on the rim.

15. A wheel is making 20 revolutions per minute. Find in radians the angle through which a spoke turns per second.

16. The arc of a circular sector is 6 cm. long and the angle of the sector is $\frac{3}{4}$ radian. What is the area of the sector?

17. One angle of a triangle is $\frac{\pi^c}{4}$, another is $\frac{\pi^c}{3}$. What is the third angle?

18. What is the complement of $\frac{\pi^c}{6}$?

19. The wheel of a carriage is 3 ft. in diameter. Through what angle (in radians) does the wheel rotate when the carriage advances 5 yards?

20. A railway line alters 57° in direction when passing round a circular arc of length $\frac{3}{4}$ mile. What is the radius of the arc, in chains, to the nearest chain?

21. A wheel of radius r feet is rotating on its axis at ω radians per second. What is the speed of a point on the rim?

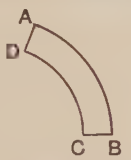


FIG. 263.

22. AB, CD are arcs of concentric circles cut off by portions of two radii AD, BC; AD = BC = 1 cm., arc AB = 5 cm., arc DC = 3 cm. Calculate the radii of the circles and the area of the figure.

23. Each point on the rim of a wheel of diameter 6 ft. has a speed of 60 m.p.h.; find the angular velocity of the wheel in radians per second.

24. In what ratio does a chord of length 8 cm. divide the circumference of a circle of diameter 10 cm. ?

25. Find from the Tables the values of (i) $\sin(1^c)$, (ii) $\sec(0 \cdot 38^c)$, (iii) $\tan(\theta^c)$, where $\theta = \sin 72°$.

26. Find, without using trigonometric tables, approximate values of :

(i) $\sin 7°$; (ii) $\sin 4° 30'$; (iii) $\sin 40'$; (iv) $\sin 36''$;
(v) $\cos 84°$; (vi) $\cos 89°$; (vii) $\tan 2° 30'$; (viii) $\cot 85°$.

27. Use the fact that, in Fig. 264, chord PQ < arc PQ < PT + TQ to show that if θ^c is acute $\sin\theta^c < \theta < \tan\theta^c$.

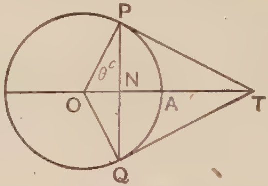


FIG. 264.

28. With the notation of No. 27, write down the length of the chord AP, and deduce that if θ^c is acute $\sin\theta^c < 2\sin\frac{\theta^c}{2} < \theta$.

29. A wheel of radius a ft. rolls, without slipping, along level ground. Initially a point P on the rim is in contact with the ground. What is the height of P above the ground when the wheel has advanced b feet ?

30. In Fig. 265, ACB is a semicircle, centre O ; its area is bisected by a line XY parallel to the diameter AB ; if $\angle AOX = \theta^c < \frac{\pi}{2}$, prove that $\frac{\pi}{2} - 2\theta = \sin 2\theta^c$. If $\frac{\pi}{2} - 2\theta = \phi$, prove that $\cos\phi^c = \phi$; verify from the Tables that $\phi^c \simeq 42° 20'$, and prove that $\angle AOX \simeq 23° 50'$.

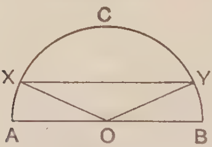


FIG. 265.

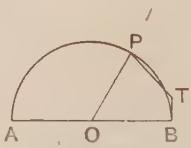


FIG. 266.

31. In Fig. 266, TB is the tangent to the circle on AB as diameter ; AB = 8 cm. ; TB = 1 cm. ; TP = 3 cm. Show that $\angle$ BOP is approximately one radian.

32. OT is a diameter of a circle, radius 5 cm.; OA, AB are arcs of the circle, each 5 cm. long; OA, OB are produced to cut the tangent at T in P, Q. Calculate PQ.

33. A fly crawls from a point A on the rim of the base of a cone of semi-vertical angle $\sin^{-1}(b)$ to the diametrically opposite point B on the rim. Show that the shortest path across the curved surface is $\dfrac{2}{\pi b}\sin\left(\dfrac{\pi b}{2}\right)^c$ times the distance along the rim.

34. A string ABC, l ft. long, is attached to the top of a circular cylinder of radius a ft., and a small heavy body is attached to C,

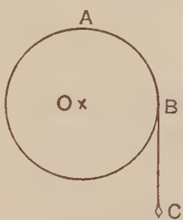


FIG. 267.

which is set swinging. (See Fig. 267.) Find an expression for the depth of C below A when the length of string in contact with the cylinder is m feet.

35. A wheel, centre A, radius a in., rolls, without slipping, in a vertical plane along the outer rim of a circular disc, centre B, radius b; initially the spoke AP is vertical. What angle does AP make with the vertical when A has moved s inches? (See Fig. 268.)

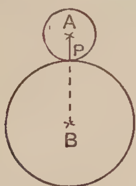


FIG. 268.

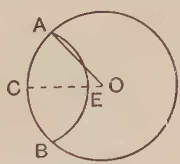


FIG. 269.

36. C is a point on a given circle, centre O; a circle AEB is drawn with centre C, so as to divide the area of the given circle in the ratio 1 : 2; if $\angle COA = \theta^c$, prove that $\sin\theta^c + (\pi - \theta)\cos\theta^c = \dfrac{2\pi}{3}$. (See Fig. 269.)

Size of a distant object.

Let AB be an object whose distance ON from O $= d$ feet.

Let AN $= h_1$ ft., NB $= h_2$ ft., $\angle$AON $= \theta_1{}^c$, $\angle$NOB $= \theta_2{}^c$.

Then $\quad h_1 = d \tan \theta_1{}^c \simeq d\theta_1$, if θ_1 is small,

$\qquad h_2 = d \tan \theta_2{}^c \simeq d\theta_2$, if θ_2 is small;

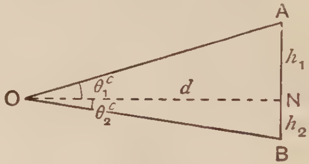


FIG. 270.

$$\therefore \text{ AB} = h_1 + h_2 \simeq d(\theta_1 + \theta_2) \text{ feet};$$

$$\therefore \text{ if } \angle \text{AOB} = \theta^c, \text{ where } \theta \text{ is small,}$$

$$\text{AB} \simeq d \,.\, \theta \text{ feet.}$$

Example III. Find the diameter of the Sun, given that its distance from the Earth is 93,000,000 miles and that it subtends an angle of 31·5′ at a point on the Earth.

From the Tables (or by calculation), $31{\cdot}5' = 0{\cdot}00915^c$;

$$\therefore \text{ diameter} = 93{,}000{,}000 \times 0{\cdot}00915 \text{ miles}$$

$$= \mathbf{850{,}000} \text{ miles.}$$

Dip of horizon.

If, from a point T above the Earth, tangents are drawn in all directions to the Earth's surface, the points of contact lie on a circle PQ, which is called the *Visible Horizon* from T. The angle ($\angle$PTH)

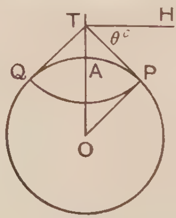


FIG. 271.

which any tangent makes with the horizontal plane through T is called the *Dip of the Horizon* and the length of TP is called the *Distance of the Horizon.*

Let the height AT of T above the Earth be h feet and let the radius of the Earth be r feet, and let $\angle HTP = \theta^c$; then $\angle TOP \doteq \angle HTP = \theta^c$.

Then $TO = r + h$ feet; $\therefore\ TP^2 = TO^2 - OP^2 = (r+h)^2 - r^2$

$$= 2rh + h^2;$$

$$\therefore\ \tan^2 \theta^c = \frac{TP^2}{OP^2} = \frac{2rh + h^2}{r^2} = 2\left(\frac{h}{r}\right) + \left(\frac{h}{r}\right)^2.$$

Now the radius of the earth (r) is about 20,000,000 feet;

$\therefore$ in practice, $\frac{h}{r}$ is always small, and $\left(\frac{h}{r}\right)^2$ will be very small compared with $\frac{h}{r}$. For example, even at a height of 5000 feet, $\frac{h}{r} \simeq \frac{1}{4000} \simeq 0{\cdot}0002$ and $\frac{h^2}{r^2} \simeq 0{\cdot}000{,}000{,}05$;

$$\therefore\ \tan \theta^c \simeq \sqrt{\left(\frac{2h}{r}\right)}$$

$$\text{and the dip} = \theta^c \simeq \sqrt{\left(\frac{2h}{r}\right)} \textbf{ radians.}$$

Further $\qquad TP = r \tan \theta^c \simeq r\theta \simeq r\sqrt{\left(\frac{2h}{r}\right)}$ feet;

$$\therefore\ \textbf{the distance of the horizon} \simeq \sqrt{\left(r^2 \times \frac{2h}{r}\right)} = \sqrt{(2hr)} \textbf{ feet.}$$

Note. This result may also be obtained as follows:

In Fig. 271, let TA cut the circle again at B.

Then $TP^2 = TA \,.\, TB \simeq TA \,.\, AB \simeq h \,.\, 2r$;

$$\therefore\ TP \simeq \sqrt{(2hr)} \text{ feet, as before.}$$

Taking the radius of the earth as 3960 miles $= 3960 \times 5280$ feet, we have

$$\sqrt{(2hr)} \text{ feet} = \sqrt{(2 \times 3960 \times 5280h)} \text{ feet} = \frac{\sqrt{(2 \times 3960 \times 5280h)}}{5280} \text{ miles}$$

$$= \sqrt{\left(\frac{2 \times 3960 \times 5280h}{5280 \times 5280}\right)} = \sqrt{\left(\frac{7920h}{5280}\right)} \text{ miles}$$

$$= \sqrt{\left(\frac{3h}{2}\right)} \text{ miles.}$$

$\therefore$ **at a height of** h ***feet*****, the distance of the horizon** $\simeq \sqrt{\left(\frac{3h}{2}\right)}$ **miles.**

For example, at a height of 150 feet, the distance visible is

$$\sqrt{\left(\frac{3 \times 150}{2}\right)} = \sqrt{(225)} = 15 \text{ miles.}$$

Approximation for $\cos\theta^c$, if θ^c is a small angle.

Draw $\triangle$CAB, so that CA = CB = 1 and $\angle$ACB = θ^c.

If CN is perpendicular to AB,

$$AB = 2AN = 2AC \sin ACN = 2\sin\frac{\theta^c}{2}.$$

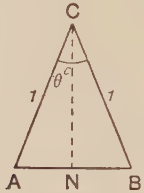


FIG. 272.

From the cosine formula, $\left(2\sin\frac{\theta^c}{2}\right)^2 = 1^2 + 1^2 - 2\cos\theta^c$;

$$\therefore\ \cos\theta^c = 1 - 2\sin^2\frac{\theta^c}{2};\ \text{but}\ \sin\frac{\theta^c}{2} \simeq \frac{\theta}{2};$$

$$\therefore\ \boldsymbol{\cos\theta^c \simeq 1 - \tfrac{1}{2}\theta^2}.$$

From the Tables, if $\theta = \frac{1}{5}$,

$\cos\frac{1}{5}^c \simeq \cos 11°\,28' \simeq 0{\cdot}9801$ and $1 - \frac{1}{2}\theta^2 = 1 - \frac{1}{50} = 0{\cdot}98$.

EXERCISE XI. b.

1. The diameter of a halfpenny is one inch. At what distance will its diameter subtend an angle of 1° ?

2. What angle does the diameter of a halfpenny subtend at the eye, if held 3 feet away ?

3. Find the diameter of the Moon if it subtends an angle of 31′ at a point on the Earth at a distance of 240,000 miles.

4. What angle does the edge of a cricket screen 9 ft. high subtend at a batsman's eye 150 yards away ?

5. The parallax of α Centauri (*i.e.* the angle subtended by the radius of the Earth's orbit) is 0·75″. What is its distance ?

6. The greatest angle which a diameter of the Earth subtends at a point on the Sun is 17·7″. Taking the radius of the Earth as 3960 miles, find the distance of the Sun.

7. [*The Gunner's Rule.*] A and B are two artillery observers at a considerable distance apart; B measures off a length BC of d feet

FIG. 273.

at right angles to AB; A observes $\angle$ BAC to be x minutes. Prove that AB is $\dfrac{d \times 1146}{x}$ yards.

8. With the data of No. 3, find at what distance a penny, diameter 1·2 inches, must be held so as just to cover the Moon.

9. The top of a chimney stack 115 ft. high sways 5 inches each side of the vertical in a strong wind. Through what total angle does the axis of the chimney swing?

10. What is the length of the side of a regular polygon of 100 sides inscribed in a circle of radius 10 yards?

11. Without using trigonometric tables, find the approximate value of (i) cos 10°, (ii) cos 50′.

12. The peak P of a mountain 8 miles away is observed from a point O to have an elevation of 3° 20′. What is the height of P above O in feet?

13. The Sun's angular diameter is 32′ 37″ when its distance is 91,400,000 miles. What is its distance when the angular diameter is only 31′ 34″?

14. Find the distance of the visible horizon from the top of a light-house 150 ft. high.

15. Calculate in minutes the dip of the horizon for a height of (i) 50 ft., (ii) 200 ft.

16. What is the height of a light-house if its light can be seen at a distance of 10 miles?

17. From a ship's masthead 80 ft. above sea-level, it is just possible to see the top of a cliff 120 ft. high. How far is the ship from the cliff?

18. An obelisk O is just visible from a ship's masthead 90 ft. high. The ship is travelling at 15 knots towards O; after what time will O be visible from the deck, which is 25 ft. above sea-level? [Take 1 nautical mile = 1·15 statute miles.]

19. From the top of Scafell, the dip of the horizon is 1°. To what height does this correspond?

20. A vibrating pendulum is inclined at θ^c to the vertical at a time t seconds after being started, where $\theta = \frac{1}{5}\sin(6t)^c$. Through

FIG. 274.

what angle does the pendulum swing, and what is the time of a complete oscillation ?

21. A particle attached to the end of a spring is executing oscillations : it moves x inches from one extreme position in t seconds, where $x = b\cos(kt)^c$, b and k being constants. The distance between its extreme positions is 6 inches, and the time of a complete oscillation (to and fro) is 0·8 sec. Find b, k.

22. At noon on a certain day the shadows of two vertical poles A, B, each 5 ft. high, are 3 ft. 3 in. and 3 ft. $1\frac{1}{2}$ in. respectively. If A is 69 miles North of B, what is the radius of the Earth according to these measurements ?

FIG. 275.

23. ABCDE is a circular arc such that arc AB = arc DE = 3 in. ; arc BC = arc CD = 5 in. ; and the angle between the tangents at A and E is 176°. Calculate the height of C above BD. (See Fig. 275.)

24. With the data of No. 23, find the difference between the arc BD and the chord BD, given that, if θ is small, $\sin\theta^c \simeq \theta - \frac{1}{6}\theta^3$.

25. The angular diameter of the Sun varies by about 1′ during the year. Find the ratio of the Earth's distances from the Sun at perihelion and aphelion, if the mean is about 32′.

26. Prove that $\sin x' \simeq \dfrac{x}{3438}$.

27. If α^c is a small angle, $\sin(\theta^c + \alpha^c) \simeq \sin\theta^c + \alpha\cos\theta^c$. Use this result to calculate sin 30° 30′, and compare your answer with the value in the Tables.

28. In Fig. 276, where CN is perpendicular to AB, show that $CN = AC\sin\frac{\theta^c}{2} = 2\sin\frac{\theta^c}{2}\cos\frac{\theta^c}{2}$, and deduce that $\sin\theta \equiv 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$.

Hence, if $\sin\theta^c \simeq \theta + a\theta^2 + b\theta^3$, where a, b are constants, show that $a=0$, $b=-\frac{1}{6}$, given that $\cos\theta^c \simeq 1 - \frac{1}{2}\theta^2$, θ^c being a small angle.

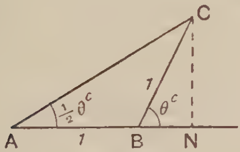


FIG. 276.

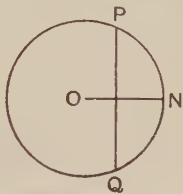


FIG. 277.

29. [*Huyghen's Formula.*] If the radius ON bisects the chord PQ, arc PQ $\simeq \frac{1}{3}$(8 chord PN − chord PQ). Prove this if ∠POQ is small; and show that the approximation even holds for

$$\sin\theta^c \simeq \theta - \tfrac{1}{6}\theta^3.$$

30. AB is the tangent at A to a circle, centre O, radius a; a chord AQ subtends an angle θ^c at O; from AB is cut off AP equal to AQ; PQ meets AO produced at R.

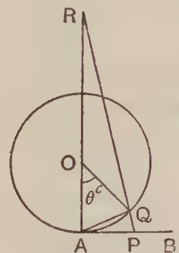


FIG. 278.

Prove that (i) $\angle APQ = \frac{\pi}{2} - \frac{\theta}{4}$ radians, (ii) $AR = 2a\sin\frac{\theta}{2}\cot\frac{\theta}{4}$.

Hence show that if θ is very small, $AR \simeq 4a$.

Graph of $\sin\theta^c$ for values of θ from 0 to π.

From the Tables we have the following values:

$\theta=0$	0·3	0·6	0·9	1·2	1·5	1·8	2·1	2·4	2·7	3·0	3·3
$\theta^c=0°$	17° 11′	34° 23′	51° 34′	68° 45′	85° 57′	103° 8′	120° 19′	137° 31′	154° 42′	171° 53′	189° 5′
$\sin\theta^c=0$	0·295	0·565	0·783	0·932	0·998	0·974	0·863	0·675	0·427	0·141	−0·158

From these values we obtain the graph shown below.

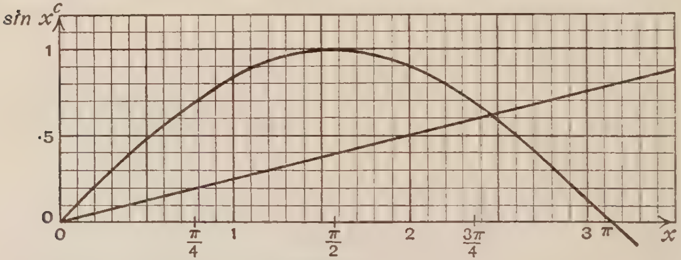


FIG. 279.

Example IV. Solve graphically $\sin x^c = \frac{1}{4}x$.

With the same scale and axes as above, draw the graph of the function $\frac{1}{4}x$.

These intersect where $x=0$ and where $x \simeq 2{\cdot}475$.

$\therefore$ the angles which satisfy the given equations are 0^c and 2·475 radians or 0° and 141° 50′ approx.

EXERCISE XI. c.

Use the graph in Fig. 279 for solving Nos. 1-15.

[Do not rule lines on the figure in the book, but use a ruler to show the position of the line when reading off a point of intersection.]

1. $\sin x^c = 0{\cdot}6$. **2.** $y = \sin 1{\cdot}3^c$. **3.** $\sin x^c = 0{\cdot}35$.

4. $y = \sin 2{\cdot}6^c$. **5.** $\cos x^c = \sin\left(\frac{\pi}{2} - x\right)^c = 0{\cdot}7$.

6. $\cos x^c = 0{\cdot}25$. **7.** $\sin x^c = \frac{1}{2}x$. **8.** $\sin x^c = \frac{1}{3}x$.

9. $\sin x^c = \frac{1}{6}x$. **10.** $\sin x^c = \frac{1}{2}x - 1$. **11.** $x + \sin x^c = 1$.

12. $\sin x^c = \frac{\pi}{2} - x$. **13.** $\cos z^c = z$.

14. $\sin\left(z+\frac{\pi}{4}\right)^c = z$. **15.** $\sin(5z)^c = z$.

16. Draw the graph of $\cos x^c$ for values of x from 0 to 3, and use it to solve the equations:

(i) $\cos x^c = x$; (ii) $\cos x^c = \frac{1}{2}x$; (iii) $\cos x^c + \frac{1}{5}x = 0$;

(iv) $\cos x^c + \frac{1}{6}x = \frac{1}{2}$; (v) $1 + \cos x^c = x$.

17. How does the graph in Fig. 279 illustrate the fact $\sin x^c < x$?

18. Find graphically a value of x, other than $x=0$, such that $\tan x^c = x$.

19. A wire 40 cm. long is bent into a buckle composed of the arc and chord of a circle of radius 10 cm. Find graphically the angle subtended at the centre of the circle by the arc.

20. In Fig. 280, the circular portion ACB of a bow is 5 ft. long and the distance CD of C from AB, where AC = CB, is 8 inches. Find graphically $\angle$ACB.

21. ACB is an arc of a circle, centre O, such that the sector AOB is three times the segment ACB. Find graphically $\angle$AOB.

22. AB is a chord of a circle, centre O, such that the area of the segment cut off by AB is one-quarter of the area of the circle. Find graphically $\angle$AOB.

FIG. 280.

23. In Fig. 281, AS, AT are tangents to a circle, including an

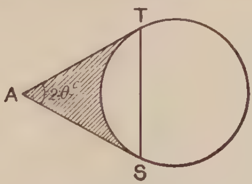


FIG. 281.

angle $2\theta^c$. Find θ if the arc TS divides the $\triangle$TAS into two portions of equal area.

24. C is the mid-point of the arc ACB of the circle, centre O; the centre of gravity of the sector AOB is a point G on OC, such that $\frac{OG}{OC} = \frac{2}{3}\frac{\sin\theta^c}{\theta}$, where $\angle AOB = 2\theta^c$. Find graphically the value of θ if G is at the mid-point of OC.

25. A thread is wound round a disc, centre O, radius a in., whose plane is vertical; initially there is a straight horizontal portion AB

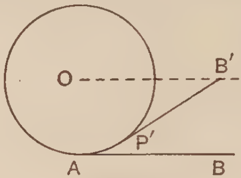


FIG. 282.

of length $2a$ in.; this is now wound into the position AP′B′, where B′ is at the same level as O. Find graphically $\angle$OB′P′.

26. *Sketch* with the same axes and scale the graph of

$$y=5\sin\frac{2\pi}{9}(x-3t)$$

for $t=0, 1, 2, 3, 4$, the unit of angle being a radian.

Suppose x and y are measured in feet and t represents seconds. Then the series of graphs represents the progress of a wave.

What is (i) the height of the crest above the trough;

(ii) the distance between successive crests;

(iii) the speed of advance of the wave?

27. Repeat No. 26 for the relation $y=a\sin\frac{2\pi}{\lambda}(x-vt)$, where a, λ, v are constants.

CHAPTER XII.

TRIANGLES AND POLYGONS.

Area of triangle.

It has been proved on p. 108 that the area Δ of any triangle ABC is given by the formula

$$\Delta = \tfrac{1}{2}ab \sin \mathrm{C} = \tfrac{1}{2}bc \sin \mathrm{A} = \tfrac{1}{2}ca \sin \mathrm{B}.$$

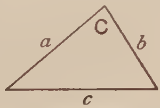


FIG. 283.

By using the cosine formula, this can be expressed in terms of the lengths of the three sides;

$$c^2 = a^2 + b^2 - 2ab \cos \mathrm{C}\,; \quad \therefore \quad \cos \mathrm{C} = \frac{a^2+b^2-c^2}{2ab}\,;$$

$$\therefore \sin^2 \mathrm{C} = 1 - \cos^2 \mathrm{C} = 1 - \left(\frac{a^2+b^2-c^2}{2ab}\right)^2$$

$$= \left(1 + \frac{a^2+b^2-c^2}{2ab}\right)\left(1 - \frac{a^2+b^2-c^2}{2ab}\right)$$

$$= \frac{2ab+a^2+b^2-c^2}{2ab} \cdot \frac{2ab-a^2-b^2+c^2}{2ab}$$

$$= \frac{(a+b)^2-c^2}{2ab} \cdot \frac{c^2-(a-b)^2}{2ab}$$

$$= \frac{(a+b+c)(a+b-c)(c+a-b)(c-a+b)}{4a^2b^2}.$$

Put $a+b+c=2s$; $\therefore\ a+b-c=2s-2c=2(s-c)$, etc.;

$$\therefore\ \Delta^2=\tfrac{1}{4}a^2b^2\sin^2 C=\frac{2s(2s-2c)(2s-2b)(2s-2a)}{16}$$

$$=s(s-a)(s-b)(s-c);$$

$$\therefore\ \Delta=\sqrt{\{s(s-a)(s-b)(s-c)\}}.$$

This result was first given by *Hero of Alexandria* about 120 B.C.; it is sometimes called Heron's formula for the area of a triangle.

Area of parallelogram.

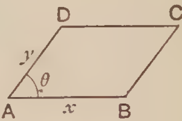


FIG. 284.

ABCD is a parallelogram, having

$AB=x$, $AD=y$, $\angle BAD=\theta$.

Then area of ABCD $=2\triangle ABD=2\times\tfrac{1}{2}xy\sin\theta$

$=\mathbf{xy\sin\theta}$.

Area of trapezium.

ABCD is a trapezium, with AB and CD as parallel sides.

Let $AB=x$, $DC=y$, and distance between AB and DC be h.

Area of ABCD $=\triangle ABD+\triangle BCD=\tfrac{1}{2}x\,.\,h+\tfrac{1}{2}y\,.\,h$

$=\tfrac{1}{2}(x+y)h$

= **half sum of parallel sides × distance between them.**

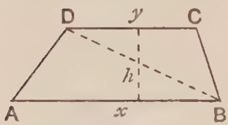


FIG. 285.

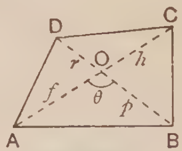


FIG. 286.

Area of quadrilateral.

ABCD is a quadrilateral, whose diagonals cut at O;

$AC=x$, $BD=y$ and $\angle AOB=\theta$.

Let $AO=f$, $OC=h$, $BO=p$, $OD=r$, so that $f+h=x$ and $p+r=y$.

Area of ABCD $=\triangle AOB+\triangle BOC+\triangle COD+\triangle DOA$

$$=\tfrac{1}{2}fp\sin\theta+\tfrac{1}{2}ph\sin(180^\circ-\theta)+\tfrac{1}{2}hr\sin\theta$$
$$+\tfrac{1}{2}rf\sin(180^\circ-\theta).$$

But $\sin(180° - \theta) = \sin\theta$;

$$\therefore \text{ area of ABCD} = \tfrac{1}{2}\sin\theta(fp + ph + hr + rf)$$
$$= \tfrac{1}{2}\sin\theta(f+h)(p+r)$$
$$= \tfrac{1}{2}\mathbf{xy}\sin\theta.$$

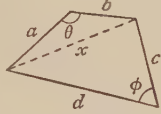


FIG. 287.

Note. If we are given the 4 sides and one angle θ of a quadrilateral, we can find the opposite angle ϕ, because

$$a^2 + b^2 - 2ab\cos\theta = x^2 = c^2 + d^2 - 2cd\cos\phi.$$

The area is then given by $\tfrac{1}{2}ab\sin\theta + \tfrac{1}{2}cd\sin\phi$.

Area of a regular n-sided polygon.

(i) Suppose the polygon is inscribed in a circle, centre O, radius R.

Let AB be one of the sides.

Then $\angle AOB = \dfrac{2\pi^c}{n}$;

$$\therefore \text{ area of } \triangle OAB = \tfrac{1}{2}R^2\sin\left(\frac{2\pi}{n}\right);$$

$$\therefore \textbf{ area of polygon} = \frac{\mathbf{nR^2}}{\mathbf{2}}\sin\left(\frac{\mathbf{2\pi}}{\mathbf{n}}\right).$$

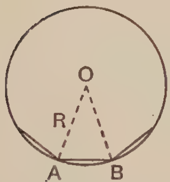


FIG. 288.

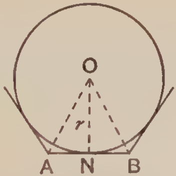


FIG. 289.

(ii) Suppose the polygon circumscribes a circle, centre O, radius r.

Let AB, one of the sides, touch the circle at N.

The $\angle AOB = \dfrac{2\pi^c}{n}$; $\therefore \angle AON = \dfrac{\pi^c}{n}$; $\therefore AN = r\tan\left(\dfrac{\pi}{n}\right)$;

$$\therefore \text{ area of } \triangle OAB = ON \,.\, AN = r^2\tan\left(\frac{\pi}{n}\right);$$

$$\therefore \textbf{ area of polygon} = \mathbf{nr^2}\tan\left(\frac{\pi}{\mathbf{n}}\right).$$

(iii) The area can also be expressed easily in terms of the length of a side.

Let $AB = a$; then $ON = \frac{a}{2}\cot\left(\frac{\pi}{n}\right)$;

$$\therefore \triangle OAB = \tfrac{1}{4}a^2 \cot\left(\frac{\pi}{n}\right);$$

$$\therefore \text{area of polygon} = \frac{na^2}{4}\cot\left(\frac{\pi}{n}\right).$$

Note. The reader should observe the form these results take when the number of sides, n, becomes very large.

In (i) we see that the area $\simeq \frac{nR^2}{2} \times \frac{2\pi}{n} \simeq \pi R^2$.

In (ii) we see that the area $\simeq nr^2 \times \frac{\pi}{n} \simeq \pi r^2$.

In (iii) we see that the area $\simeq \frac{na^2}{4} \times \frac{n}{\pi} \simeq \frac{n^2a^2}{4\pi}$

$$\simeq \frac{(\text{perimeter})^2}{4\pi}.$$

EXERCISE XII. a.

Calculate the areas of the following figures, Nos. 1-9.

1.

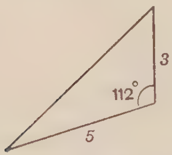


FIG. 290.

2.

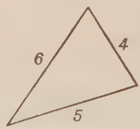


FIG. 291.

3

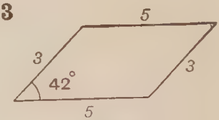


FIG. 292.

4.

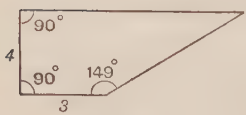


FIG. 293.

5.

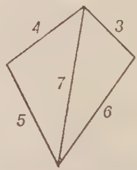


FIG. 294.

6.

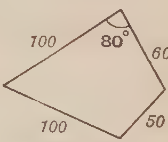


FIG. 295.

7.

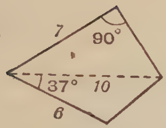


FIG. 296.

8.

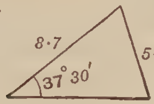


FIG. 297

9.

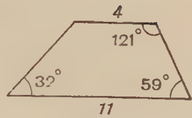


FIG. 298.

10. Two sides of a triangle are 51 yd. and 92 yd., and the area is 980 sq. yd. What can you say about the included angle ?

11. The parallel sides of a trapezium are 2, 5 in. ; the oblique sides are 4, 6 in. Find the angles and the area.

12. Calculate the acute angle of a rhombus whose area is 1 sq. ft., if the length of each side is 15 inches.

13. A rectangle, with base 5 inches, is constructed equal in area to an equilateral triangle of side 6 inches. What is the height of the rectangle ?

14. On a map, scale 2 inches to the mile, a plot of ground is represented by a triangle ABC, where $AB = 1{\cdot}3$ in., $BC = 2{\cdot}1$ in., $\angle ABC = 117°$. Find its area in acres.

15. One side of a parallelogram is 6 cm., one angle is 141° ; the area is 27 sq. cm. Find the other side.

16. A piece of wire 6 ft. long is bent to form a regular 9-sided polygon. What is its area ?

17. A parallelogram with sides 4 cm., 6 cm. and one angle 113° is equal in area to a parallelogram with sides 5 cm., 7 cm. Find the acute angle of the latter.

18. ABCD is a quadrilateral ; $AB=4$, $BC=2$, $CD=3$, $\angle ABC=122°$, $\angle BCD=157°$. Find the area.

19. A regular 7-sided polygon is inscribed in a circle of radius 5 cm. Find its area.

20. A regular pentagon is formed from a given length of flexible wire. What is the percentage increase of area if the same piece of wire is bent to form a regular decagon ?

21. A piece of wire $2\frac{1}{2}$ ft. long is bent to form a regular pentagon. What is the radius of the circle which (i) circumscribes it, (ii) is inscribed in it ?

22. The diagonals of a quadrilateral are 7 in., 6 in. long, and its area is 10 sq. in. At what angle do the diagonals cut ?

23. A " Soccer " goal is 8 yd. wide, 8 ft. high ; the goal line runs East and West. Find the area enclosed between the shadows of the goal posts and cross-bar and the goal line when the Sun is S.W. at elevation 33°.

24. A piece of wire 5 ft. long is bent into a triangle; two of the angles are 108°, 47°. Find the area.

25. A piece of ground on a sloping hill-side has an area of 2·175 square miles. On a map it is shown as an area of 1·942 miles. At what angle is the hill-side inclined to the horizontal?

26. In Fig. 299, ABCD is a rectangle. Find PQ and the area of AQCP in terms of l, θ.

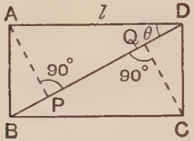


FIG. 299.

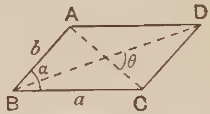


FIG. 300.

27. In Fig. 300, ABCD is a parallelogram. Prove that

$$\tan\theta = \frac{2ab\sin\alpha}{a^2 - b^2}.$$

28. In Fig. 301, $\angle$ACB$=90°$ and AP$=$PB. Prove that

$$\sin\theta = \frac{2ab}{c^2}.$$

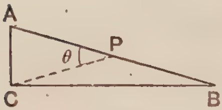


FIG. 301.

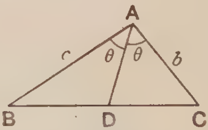


FIG. 302.

29. In Fig. 302, AD bisects $\angle$ BAC. Prove that AD $= \dfrac{bc\sin 2\theta}{(b+c)\sin\theta}$, and use the relation $\sin 2\theta = 2\sin\theta\cos\theta$ (Ex. XI. b, No. 28) to simplify the expression.

Pyramids.

If the base of a pyramid is a *regular* polygon, and if the perpendicular from the vertex to the base passes through the centre of the regular polygon, the solid is called a *right pyramid*.

It has already been mentioned (p. 153) that the volume of *any* pyramid is measured by $\frac{1}{3}$ base-area × height.

Sections of a pyramid parallel to the base are the same shape as the base, and therefore the ratio of the areas of any two such sections equals the square of the ratio of corresponding sides *or* the square of

the ratio of the distances of the sections from the vertex of the pyramid.

The volume of any frustum of a pyramid may be obtained, as on pp. 153-154, by completing the pyramid.

The argument, used on p. 154 for the frustum of a cone, shows also that if s_1, s_2 are the areas of the parallel faces of a frustum of a pyramid, and if h is the distance between them, *the volume of the frustum is* $\frac{h}{3}(s_1+\sqrt{s_1 s_2}+s_2)$.

The slant faces of a pyramid are triangles, and their areas may therefore be obtained by using the ordinary triangle formulae.

EXERCISE XII. b.

1. A right pyramid 6 cm. high stands on a square base of side 16 cm. Calculate (i) its volume, (ii) the area of its total surface.

2. A right pyramid vertex O stands on a square base ABCD ; AB=5 in. ; $\angle$AOB=50°. What is the volume of the pyramid ?

3. The base area of a pyramid is 80 sq. cm. and its height is 12 cm. Find (i) its volume, (ii) the area of a section parallel to the base and 3 cm. from the base, (iii) the volume of the frustum bounded by the base and a plane parallel to the base and 3 cm. from it.

4. A frustum of a pyramid is bounded by two rectangles 6 in. by 8 in. and 9 in. by 12 in. at a distance 5 in. apart. What is its volume ?

5. If, with the data of No. 4, all the slant edges are equal, find the total area of the surface.

6. A chimney 40 ft. high tapers uniformly, the base being 12 ft. square and the top 10 ft. square ; the central hollow space has a uniform circular section, 4 ft. in diameter. Find to the nearest ton the weight of brick-work, if 16 cu. ft. weigh a ton.

7. Fig. 303 represents in plan a stack on a rectangular base ; the ridge EF is 15 ft. above the base ABCD. Find the volume of the stack.

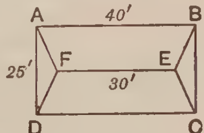


FIG. 303.

8. Find the volume of a regular tetrahedron (a pyramid on a triangular base) if each edge is 4 inches.

9. A pyramid, vertex O, stands on a rectangular base ABCD ; the slant edges are all equal ; $\angle$AOB=37°, AB=6 cm., BC=8 cm. Find its volume.

10. A pyramid stands on a square horizontal base ; each face makes an angle α with the vertical ; each edge makes an angle β with the vertical. Find a relation connecting α and β.

Radius of the circumcircle of $\triangle$ABC.

Let O be the circumcentre and R the length of the circumradius of $\triangle$ABC, and let CO meet the circumcircle at P.

Then $\angle$CBP$=90°$, $\angle$ in semicircle.

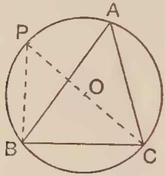


FIG. 304.

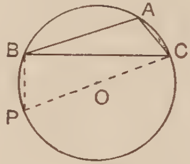


FIG. 305.

Also $\angle$BPC$=\angle$BAC$=$A, in Fig. 304,

and $\angle$BPC$=180°-\angle$BAC$=180°-$A, in Fig. 305.

$\therefore\ a=$BC$=$CP sin BPC$=2$R sin A (Fig. 304)

$=2$R sin $(180°-$A$)$ (Fig. 305).

$\therefore$ in each case, $a=2\text{R}\sin\text{A}$;

$$\therefore\ \mathbf{R}=\frac{\mathbf{a}}{\mathbf{2\,sin\,A}}.$$

Similarly,
$$\mathbf{R}=\frac{\mathbf{b}}{\mathbf{2\,sin\,B}}=\frac{\mathbf{c}}{\mathbf{2\,sin\,C}}.$$

Note. Since $\Delta=\frac{1}{2}bc\sin\text{A}=\frac{1}{2}bc\,.\,\frac{a}{2\text{R}}$,

$$\therefore\ \Delta=\frac{abc}{4\text{R}}\quad\text{and}\quad \text{R}=\frac{abc}{4\Delta}.$$

Radius of the inscribed circle of $\triangle$ABC.

Let I be the centre and r the length of the radius of the inscribed circle, which touches the sides at X, Y, Z.

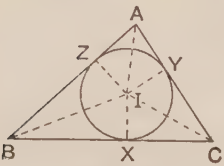


FIG. 306.

Then $\Delta=$area of IBC$+$area of ICA$+$area of IAB

$=\frac{1}{2}$IX . BC$+\frac{1}{2}$IY . CA$+\frac{1}{2}$IZ . AB

$=\frac{1}{2}ra+\frac{1}{2}rb+\frac{1}{2}rc=\frac{1}{2}r(a+b+c)$;

put $a+b+c=2s$,

$=\frac{1}{2}r\,.\,2s=rs$;

$$\therefore\ \mathbf{r}=\frac{\Delta}{\mathbf{s}}.$$

Radius of an escribed circle of $\triangle$ABC.

Let I_1 be the centre and r_1 the length of the radius of the circle escribed to BC, which touches BC, CA, AB at X_1, Y_1, Z_1.

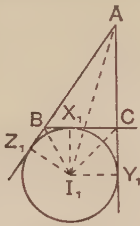


FIG. 307.

Then $\quad \Delta$ = area of I_1CA + area of I_1AB − area of I_1BC

$$= \tfrac{1}{2}I_1Y_1 \,.\, CA + \tfrac{1}{2}I_1Z_1 \,.\, AB - \tfrac{1}{2}I_1X_1 \,.\, BC$$

$$= \tfrac{1}{2}r_1 b + \tfrac{1}{2}r_1 c - \tfrac{1}{2}r_1 a = \tfrac{1}{2}r_1(b+c-a).$$

But if $a+b+c=2s$, $b+c-a=2s-2a$,

$$\therefore\ \Delta = \tfrac{1}{2}r_1(2s-2a) = r_1(s-a);$$

$$\therefore\ \mathbf{r_1 = \frac{\Delta}{s-a}}.$$

Similarly, if r_2, r_3 are the radii of the circles escribed to CA, AB,

$$\mathbf{r_2 = \frac{\Delta}{s-b}} \quad \text{and} \quad \mathbf{r_3 = \frac{\Delta}{s-c}}.$$

Note. (i) Since IB, IC bisect the angles at B, C,

$$BX = r \cot\frac{B}{2} \quad \text{and} \quad XC = r\cot\frac{C}{2};$$

$$\therefore\ r\left(\cot\frac{B}{2} + \cot\frac{C}{2}\right) = BX + XC = BC = a.$$

(ii) Since I_1B, I_1C bisect the exterior angles at B, C,

$$\angle I_1BX_1 = \tfrac{1}{2}(180^\circ - B) = 90^\circ - \frac{B}{2}; \quad \therefore\ \angle BI_1X_1 = \frac{B}{2};$$

$$\therefore\ BX_1 = r_1 \tan\frac{B}{2} \quad \text{and} \quad X_1C = r_1\tan\frac{C}{2};$$

$$\therefore\ r_1\left(\tan\frac{B}{2} + \tan\frac{C}{2}\right) = BX_1 + X_1C = BC = a.$$

These results express r, r_1, etc., in terms of one side, and two angles of the triangle.

The positions of the points of contact of the in-circle and ex-circle are obtained from the following results :

(i) In Fig. 306, $AY=AZ=s-a$; $BZ=BX=s-b$; $CX=CY=s-c$.

Since the tangents from a point to a circle are equal, $AY=AZ$, $BZ=BX$, $CX=CY$;

$$\therefore\ AY+BX+XC=\text{semi-perimeter}=s.$$

But $\quad BX+XC=BC=a$; $\quad \therefore\ \mathbf{AY=s-a}$,

and similarly for the other tangents.

(ii) In Fig. 307, $AY_1=AZ_1=s$; $BZ_1=BX_1=s-c$; $CX_1=CY_1=s-b$.

$$AY_1+AZ_1=AB+BZ_1+AC+CY_1=AB+BX_1+AC+CX_1=2s;$$

$$\therefore\ \mathbf{AY_1=AZ_1=s};$$

$$\therefore\ BZ_1=AZ_1-AB=s-c, \quad \text{and similarly for } CY_1.$$

Note. These results give alternative forms for r, r_1.

Thus $\qquad r=IX=BX\tan\frac{B}{2}=(s-b)\tan\frac{B}{2}$, etc.,

$$r_1=I_1X_1=BX_1\cot\frac{B}{2}=(s-c)\cot\frac{B}{2}, \text{ etc.}$$

EXERCISE XII. c.

1. Find the radius of a circle if a chord 3·6 in. long subtends an angle of 113° at the circumference.

2. Find the radius of a circle if a chord 5·72 in. long subtends an angle of (i) 52°, (ii) 128° at the circumference.

3. Find R and r in a triangle whose sides are 5, 6, 7 inches.

4. Find the radius of each escribed circle of the triangle whose sides are 3, 4, 5 inches.

5. In $\triangle ABC$, $b=c=6$, $A=50°$. Find R and r.

6. In $\triangle ABC$, $a=3$, $B=57°$, $C=42°$. Find R and r.

7. Given $R=14$ cm., $a=12$ cm. Find A.

8. Three places A, B, C are each 4 miles distant from a place O. If the angle ABC is 71°, find the distance of A from C.

9. The cross-section of a long prism is a triangle with sides 8, 9, 11 cm. long. What is the internal diameter of the smallest cylindrical pipe through which it can be passed ?

10. ABCD is a quadrilateral ; $BA=4$ in., $AD=5$ in., $\angle BAD=41°$; $\angle ABC=\angle ADC=90°$. Find AC.

11. In Fig. 306, find BX and CX, (i) if $a=5''$, $b=6''$, $c=7''$; (ii) if $a=14$ cm., $b=20$ cm., $c=30$ cm.

12. In Fig. 307, find BX_1 and AZ_1 with the measurements of No. 11.

13. The radii of three circles, centres P, Q, R, which touch each other externally, are a, b, c inches. Find expressions for (i) the area, (ii) the radii of the inscribed and escribed circles of the triangle PQR.

14. A bracket consists of three rods forming a triangle of sides 6, 9, 11 inches, fixed in a horizontal plane, a sphere of diameter 8 inches rests on the bracket. Find the height of the highest point of the sphere above the bracket.

15. A line is drawn through the vertex A of a triangle ABC to meet BC at D. Show that the ratio of the radii of the circumcircles of triangles ABD, ACD is $\frac{c}{b}$.

Prove that the radius of a circle through A and touching BC at C is $\frac{b}{2 \sin C}$.

16. Prove that for an equilateral triangle, $r_1 = 3r$.

17. Prove that $rr_1 = \Delta \tan \frac{A}{2}$.

18. Prove that $rr_1r_2r_3 = \Delta^2$.

19. Prove that $2R^2 \sin A \sin B \sin C = \Delta$.

20. Prove that $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}$.

21. Prove that $II_1 = a \sec \frac{A}{2}$.

22. Prove that $r_1r_2 + r_1r_3 = as$.

REVISION PAPERS. R. 27-34.

R. 27.

1. The continuous line in Fig. 308 shows a section of some corrugated iron. The curve is formed of equal arcs of circles and the centre O of the first arc is 4″ below the straight line AB. Find the

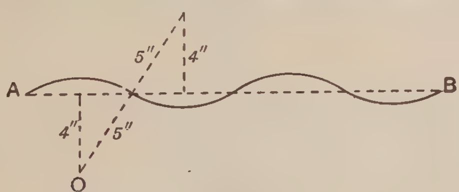


FIG. 308.

length of the curved line shown in the figure. What would be the full width of the corrugated iron covering a roof 10 ft. wide, if it were beaten out flat ?

2. An ink-bottle is in the form of a cylinder with a large conical opening. When it is filled level with the bottom of the opening, it can just be turned upside down without any ink spilling. Prove that the depth of the cone $=\frac{3}{5}$ depth of the whole bottle. (See Fig. 309.)

3. Find the other sides of a triangle in which $a=14{\cdot}7$ cm. $A=72°\ 30'$, $B=7°\ 42'$.

FIG. 309.

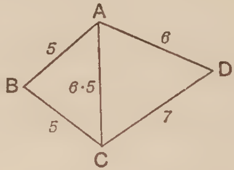


FIG. 310.

4. In a range-finder, mirrors 30 in. apart are focussed on the object whose range is being taken. At what angle will the mirrors be inclined to each other when the range of an object 600 yds. away is taken?

5. Find the area of the quadrilateral ABCD in Fig. 310.

R. 28.

1. AB and CD are two diameters at right angles of a circle, radius 10 cm.; arcs are drawn as in Fig. 311, with A and B as their centres. Calculate the perimeter and area of the shaded portion.

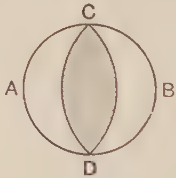


FIG. 311.

2. A tower at a distance of 1 mile subtends an angle of $30'\ 36''$. Find its height approximately.

3. Show that, if θ is measured in radians and is small, $\dfrac{3\sin\theta}{2+\cos\theta}$ is approximately equal to θ. Test this result when $\theta=0{\cdot}1$, and show they agree to three figures.

4. A country road goes direct from X to Y. XW and WY are main roads, see Fig. 312.

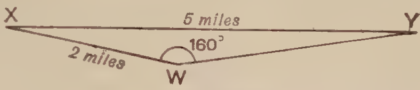


FIG. 312.

If a man can motor at 20 m.p.h. along the country road and at 30 m.p.h. along the main roads, find which is the quicker route from X to Y, and how much time he will save by taking it.

5. Regular 20-sided polygons are inscribed in and circumscribed to a circle of radius 10 cm.

Express their areas as percentages of the area of the circle.

R. 29.

1. Apply the approximate rule, that the area of a small segment of a circle $=\frac{2}{3}$ base × height, to the minor segment cut off by a chord equal to the radius of the circle. Show that in this instance the rule is equivalent to taking π equal to $4-\frac{1}{2}\sqrt{3}$, and find the error per cent. to one significant figure.

2. A base-line AB is 40 chains long. A point P is observed in the same horizontal plane, and it is found that

$$\angle PAB = 76° 48', \quad \angle ABP = 58° 32'.$$

Find the distances of P from A and from B.

3. A, B, C are three bullet-marks on a target. AB = 1 in., BC = 0·8 in., CA = 0·7 in. Find the diameter of the 'group' that they form, *i.e.* the diameter of the smallest circle into which they will just fit.

4. Find the area of a triangle ABC in which

$$a = 47{\cdot}56 \text{ in.}, \quad b = 20{\cdot}78 \text{ in.}, \quad C = 68° 43'.$$

5. A right pyramid, height 6″, stands on a square base, side 5″. Find the *total* surface area of the pyramid and the inclination of the edges to the plane of the base.

R. 30.

1. A box with a square section ABCD, side 2′, is rolled along the ground, turning in succession about the corners A, B, C, D. Sketch the path followed by the corner D until the side AD is again on the ground, and calculate its length. (See Fig. 313.)

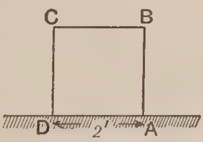


FIG. 313.

2. An arc of a circle is 2 ft. long and subtends an angle of 25° 30′ at the centre of the circle. Find the radius of the circle and the length of the chord of the arc.

3. A man is at a place in Lat. 51° N. and motors 100 miles due North. What latitude is he then in ?

4. An octagonal tower whose sides are 8 ft. ends in a pyramid whose faces slope at 70° to the vertical. Find the height and volume of the pyramid.

5. A cone is such that its curved surface when laid out flat makes an exact semicircle of radius a. What is the length of the shortest distance across the curved surface of the cone between two points at opposite ends of a diameter of the base of the cone ?

R. 31.

1. In Fig. 314, AB = BC = CD = 2 in. ; semicircles are drawn as shown. Prove that the shaded area is one-third of the area of the whole circle.

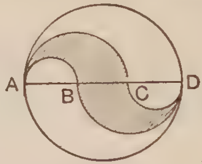


FIG. 314.

2. The gradient of a railway changes from 1 in 40 to 1 in 45. Find approximately the change in the angle of slope.

3. Find the four parallels of latitude, two N. of the Equator and two S. of it, which divide the Earth's surface into 5 zones of equal area.

4. Fig. 315 is cut out in cardboard and the four congruent triangles are folded about the sides of the square to form a pyramid. Find the total surface-area and the volume of the pyramid so formed.

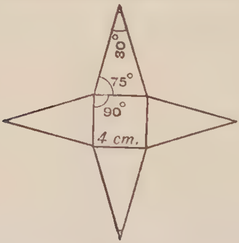


FIG. 315.

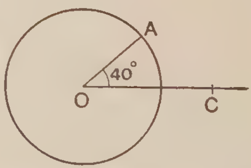


FIG. 316.

5. In Fig. 316, O is the centre of a circle, radius 3 cm., and OC = 5 cm. Calculate the radius of a circle which touches this circle at A and passes through C.

R. 32.

1. In Fig. 317, AB is a diameter; AP $=8$ cm., BQ $=4\cdot5$ cm. Calculate $\angle$ARQ.

2. With the data of No. 1, calculate the length of the arc PB and the area bounded by the arc PB and the straight lines AP, AB.

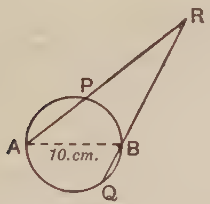


FIG. 317.

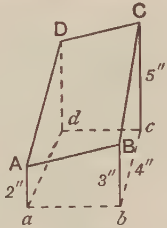


FIG. 318.

3. In the triangle ABC, AC $=25$, BC $=20$, $\angle$ BAC $=40°$. Calculate $\angle$ACB.

4. The discrepancy between observation and theory (Newtonian) of the rotation of the axis of Mercury's orbit is about 42 seconds of angle per century. At what distance does a halfpenny (diameter 1″) subtend this angle?

5. Fig. 318 represents a prism on a square horizontal base *abcd* with vertical edges *a*A, *b*B, *c*C, *d*D; the prism is cut by a plane in a section ABCD, which is therefore a parallelogram. Calculate $\angle$ABC, and find the area of ABCD. Hence deduce the angle between the planes ABCD and *abcd*.

R. 33.

1. A conical funnel, vertex O, vertical angle 50°, is suspended from a point A on the rim of the base; G is a point on the axis OC of the cone, such that OG $=\frac{2}{3}$ OC. (See Fig. 319.)

If the funnel rests with G vertically below A, find the angle which AO makes with the vertical.

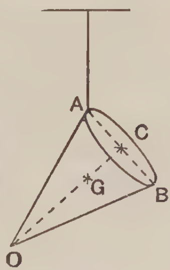


FIG. 319.

2. If a regular pentagon and a regular decagon, sides p cm., d cm. respectively, are inscribed in a circle of radius a cm., then $p^2=a^2+d^2$. Use Tables to verify this result.

3. The ends B, C of two rods AB, AC are joined by a stretched elastic string AB $=2$ ft., AC $=3$ ft. Initially BC is 1 ft. 6 in.; through what further angle must the rods be opened to cause an additional extension of 6 inches in the string?

4. What is the height of a light-house above sea-level, if its flash is visible at a distance of 12 miles ?

5. In $\triangle ABC$, $\angle ABC = 33°$, $\angle ACB = 65°$, $BC = 5$ cm. Calculate the radius of the circle, escribed to BC.

R. 34.

1. A rectangular block rests, with one edge through A on the ground, across a cylinder of diameter 10 cm. also on the ground.

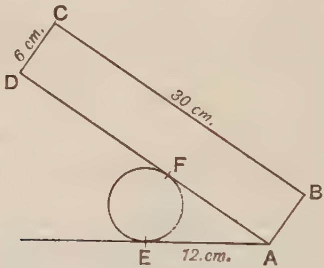


FIG. 320.

Find the height of C above the ground and the distance of C from the vertical plane through the axis of the cylinder.

2. With the data of No. 1, calculate the length of the minor arc EF.

3. AB is a diameter and AC is a chord of a circle; E is the mid-point of AC; $AB = 3$ in., $AC = 2$ in. Calculate $\angle ABE$.

4. In Fig. 321, AE is perpendicular to BC; $AE = h$, $\angle ABC = \theta°$. Express the radius of the circle in terms of h, θ.

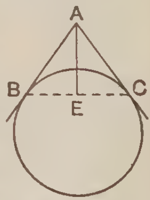


FIG. 321.

5. A pyramid has a square base with 4 equal isosceles triangles for faces. Prove that if each of these faces makes an angle of 45° with the base, the cosine of the vertical angle of each isosceles triangle will be $\frac{1}{3}$, and that the angle between a pair of triangular faces will be 120°.

NOTE ON TABLES

Usually the "Difference Columns" give average differences, calculated over intervals of 1°; but, if these differences are changing rapidly, the error introduced by taking the average over so large an interval becomes serious, and it is necessary to use a smaller interval. Accordingly, where necessary, the Tables give the average Difference for 1′, calculated over 12′ intervals.

Example. Find tan 75° 56′ and tan 75° 58′.

	$\tan 75°\,54' \simeq 3{\cdot}9812$	Diff. for 1′, interval 49 to 59, is 49;
Add	Diff. for 2′ 98	
	$\therefore \tan 75°\,56' \simeq \mathbf{3{\cdot}9910}$	∴ Diff. for 2′ is $49 \times 2 = 98$.

	$\tan 75°\,54' \simeq 3{\cdot}9812$	OR	$\tan 76°\,0' \simeq 4{\cdot}0108$
Add	Diff. for 4′ 196	*Subtract*	Diff. for 2′ 98
	$\therefore \tan 75°\,58' \simeq \mathbf{4{\cdot}0008}$		$\therefore \tan 75°\,58' \simeq \mathbf{4{\cdot}0010}$

Example. Find cosec 4° 32′.

	$\operatorname{cosec} 4°\,30' \simeq 12{\cdot}75$	Diff. for 1′, interval 25 to 35, is 5;
Subtract.	Diff. for 2′ 10	
	$\therefore \operatorname{cosec} 4°\,32' \simeq \mathbf{12{\cdot}65}$	∴ Diff. for 2′ is $5 \times 2 = 10$.

D.W.T. *

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	·0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	·0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	·0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	·1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	·1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	·1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	·2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	·2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	·2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	·2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	·3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	·3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	·3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	·3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	·3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	·3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	·4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	·4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	·4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	·4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	·4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	·4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	·5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	·5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	·5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	·5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	·5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	·5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	·5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	·5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	·6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	·6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	·6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	·6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	·6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	·6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	·6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	·6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	·6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	·6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	·6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	·7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	·7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	·7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	·7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
55	·7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	·7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	·7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	·7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	·7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	·7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	·7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	·7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	·7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	·8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	·8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	·8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	·8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	·8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	·8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	·8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	·8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	·8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	·8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	·8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	·8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	·8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	·8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	·8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	·8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	·9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	·9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	·9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	·9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	·9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	·9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	·9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	·9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	·9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	·9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	·9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	·9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	·9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	·9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	·9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	·9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	·9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	·9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	·9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	·9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Difference for 1′				
											1 to 11	13 to 23	25 to 35	37 to 47	49 to 59
0°	$-\infty$	$\bar{3}$·242	$\bar{3}$·543	$\bar{3}$·719	$\bar{3}$·844	$\bar{3}$·941	$\bar{2}$·020	$\bar{2}$·087	$\bar{2}$·145	$\bar{2}$·196					
1	$\bar{2}$·2419	2832	3210	3558	3880	4179	4459	4723	4971	5206	35	32	29	27	25
2	$\bar{2}$·5428	5640	5842	6035	6220	6397	6567	6731	6889	7041	23	22	21	20	19
3	$\bar{2}$·7188	7330	7468	7602	7731	7857	7979	8098	8213	8326					
4	$\bar{2}$·8436	8543	8647	8749	8849	8946	9042	9135	9226	9315	18	17	16	15	15
5	$\bar{2}$·9403	9489	9573	9655	9736	9816	9894	9970	*0046*	*0120*	14	14	13	13	12
6	$\bar{1}$·0192	0264	0334	0403	0472	0539	0605	0670	0734	0797	12	11	11	11	10
											1′	2′	3′	4′	5′
7	$\bar{1}$·0859	0920	0981	1040	1099	1157	1214	1271	1326	1381	10	19	29	38	48
8	$\bar{1}$·1436	1489	1542	1594	1646	1697	1747	1797	1847	1895	8	17	25	34	42
9	$\bar{1}$·1943	1991	2038	2085	2131	2176	2221	2266	2310	2353	8	15	23	30	38
10	$\bar{1}$·2397	2439	2482	2524	2565	2606	2647	2687	2727	2767	7	14	20	27	34
11	$\bar{1}$·2806	2845	2883	2921	2959	2997	3034	3070	3107	3143	6	12	19	25	31
12	$\bar{1}$·3179	3214	3250	3284	3319	3353	3387	3421	3455	3488	6	11	17	23	28
13	$\bar{1}$·3521	3554	3586	3618	3650	3682	3713	3745	3775	3806	5	11	16	21	26
14	$\bar{1}$·3837	3867	3897	3927	3957	3986	4015	4044	4073	4102	5	10	15	20	24
15	$\bar{1}$·4130	4158	4186	4214	4242	4269	4296	4323	4350	4377	5	9	14	18	23
16	$\bar{1}$·4403	4430	4456	4482	4508	4533	4559	4584	4609	4634	4	9	13	17	21
17	$\bar{1}$·4659	4684	4709	4733	4757	4781	4805	4829	4853	4876	4	8	12	16	20
18	$\bar{1}$·4900	4923	4946	4969	4992	5015	5037	5060	5082	5104	4	8	11	15	19
19	$\bar{1}$·5126	5148	5170	5192	5213	5235	5256	5278	5299	5320	4	7	11	14	18
20	$\bar{1}$·5341	5361	5382	5402	5423	5443	5463	5484	5504	5523	3	7	10	14	17
21	$\bar{1}$·5543	5563	5583	5602	5621	5641	5660	5679	5698	5717	3	6	10	13	16
22	$\bar{1}$·5736	5754	5773	5792	5810	5828	5847	5865	5883	5901	3	6	9	12	15
23	$\bar{1}$·5919	5937	5954	5972	5990	6007	6024	6042	6059	6076	3	6	9	12	15
24	$\bar{1}$·6093	6110	6127	6144	6161	6177	6194	6210	6227	6243	3	6	8	11	14
25	$\bar{1}$·6259	6276	6292	6308	6324	6340	6356	6371	6387	6403	3	5	8	11	13
26	$\bar{1}$·6418	6434	6449	6465	6480	6495	6510	6526	6541	6556	3	5	8	10	13
27	$\bar{1}$·6570	6585	6600	6615	6629	6644	6659	6673	6687	6702	2	5	7	10	12
28	$\bar{1}$·6716	6730	6744	6759	6773	6787	6801	6814	6828	6842	2	5	7	9	12
29	$\bar{1}$·6856	6869	6883	6896	6910	6923	6937	6950	6963	6977	2	4	7	9	11
30	$\bar{1}$·6990	7003	7016	7029	7042	7055	7068	7080	7093	7106	2	4	6	9	11
31	$\bar{1}$·7118	7131	7144	7156	7168	7181	7193	7205	7218	7230	2	4	6	8	10
32	$\bar{1}$·7242	7254	7266	7278	7290	7302	7314	7326	7338	7349	2	4	6	8	10
33	$\bar{1}$·7361	7373	7384	7396	7407	7419	7430	7442	7453	7464	2	4	6	8	10
34	$\bar{1}$·7476	7487	7498	7509	7520	7531	7542	7553	7564	7575	2	4	6	7	9
35	$\bar{1}$·7586	7597	7607	7618	7629	7640	7650	7661	7671	7682	2	4	5	7	9
36	$\bar{1}$·7692	7703	7713	7723	7734	7744	7754	7764	7774	7785	2	3	5	7	9
37	$\bar{1}$·7795	7805	7815	7825	7835	7844	7854	7864	7874	7884	2	3	5	7	8
38	$\bar{1}$·7893	7903	7913	7922	7932	7941	7951	7960	7970	7979	2	3	5	6	8
39	$\bar{1}$·7989	7998	8007	8017	8026	8035	8044	8053	8063	8072	2	3	5	6	8
40	$\bar{1}$·8081	8090	8099	8108	8117	8125	8134	8143	8152	8161	1	3	4	6	7
41	$\bar{1}$·8169	8178	8187	8195	8204	8213	8221	8230	8238	8247	1	3	4	6	7

Where the integer changes, the numbers are italicised.

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
42°	$\bar{1}$·8255	8264	8272	8280	8289	8297	8305	8313	8322	8330	1	3	4	6	7
43	$\bar{1}$·8338	8346	8354	8362	8370	8378	8386	8394	8402	8410	1	3	4	5	7
44	$\bar{1}$·8418	8426	8433	8441	8449	8457	8464	8472	8480	8487	1	3	4	5	6
45	$\bar{1}$·8495	8502	8510	8517	8525	8532	8540	8547	8555	8562	1	2	4	5	6
46	$\bar{1}$·8569	8577	8584	8591	8598	8606	8613	8620	8627	8634	1	2	4	5	6
47	$\bar{1}$·8641	8648	8655	8662	8669	8676	8683	8690	8697	8704	1	2	3	5	6
48	$\bar{1}$·8711	8718	8724	8731	8738	8745	8751	8758	8765	8771	1	2	3	4	6
49	$\bar{1}$·8778	8784	8791	8797	8804	8810	8817	8823	8830	8836	1	2	3	4	5
50	$\bar{1}$·8843	8849	8855	8862	8868	8874	8880	8887	8893	8899	1	2	3	4	5
51	$\bar{1}$·8905	8911	8917	8923	8929	8935	8941	8947	8953	8959	1	2	3	4	5
52	$\bar{1}$·8965	8971	8977	8983	8989	8995	9000	9006	9012	9018	1	2	3	4	5
53	$\bar{1}$·9023	9029	9035	9041	9046	9052	9057	9063	9069	9074	1	2	3	4	5
54	$\bar{1}$·9080	9085	9091	9096	9101	9107	9112	9118	9123	9128	1	2	3	4	5
55	$\bar{1}$·9134	9139	9144	9149	9155	9160	9165	9170	9175	9181	1	2	3	3	4
56	$\bar{1}$·9186	9191	9196	9201	9206	9211	9216	9221	9226	9231	1	2	3	3	4
57	$\bar{1}$·9236	9241	9246	9251	9255	9260	9265	9270	9275	9279	1	2	2	3	4
58	$\bar{1}$·9284	9289	9294	9298	9303	9308	9312	9317	9322	9326	1	2	2	3	4
59	$\bar{1}$·9331	9335	9340	9344	9349	9353	9358	9362	9367	9371	1	1	2	3	4
60	$\bar{1}$·9375	9380	9384	9388	9393	9397	9401	9406	9410	9414	1	1	2	3	4
61	$\bar{1}$·9418	9422	9427	9431	9435	9439	9443	9447	9451	9455	1	1	2	3	3
62	$\bar{1}$·9459	9463	9467	9471	9475	9479	9483	9487	9491	9495	1	1	2	3	3
63	$\bar{1}$·9499	9503	9506	9510	9514	9518	9522	9525	9529	9533	1	1	2	3	3
64	$\bar{1}$·9537	9540	9544	9548	9551	9555	9558	9562	9566	9569	1	1	2	2	3
65	$\bar{1}$·9573	9576	9580	9583	9587	9590	9594	9597	9601	9604	1	1	2	2	3
66	$\bar{1}$·9607	9611	9614	9617	9621	9624	9627	9631	9634	9637	1	1	2	2	3
67	$\bar{1}$·9640	9643	9647	9650	9653	9656	9659	9662	9666	9669	1	1	2	2	3
68	$\bar{1}$·9672	9675	9678	9681	9684	9687	9690	9693	9696	9699	0	1	1	2	2
69	$\bar{1}$·9702	9704	9707	9710	9713	9716	9719	9722	9724	9727	0	1	1	2	2
70	$\bar{1}$·9730	9733	9735	9738	9741	9743	9746	9749	9751	9754	0	1	1	2	2
71	$\bar{1}$·9757	9759	9762	9764	9767	9770	9772	9775	9777	9780	0	1	1	2	2
72	$\bar{1}$·9782	9785	9787	9789	9792	9794	9797	9799	9801	9804	0	1	1	2	2
73	$\bar{1}$·9806	9808	9811	9813	9815	9817	9820	9822	9824	9826	0	1	1	1	2
74	$\bar{1}$·9828	9831	9833	9835	9837	9839	9841	9843	9845	9847	0	1	1	1	2
75	$\bar{1}$·9849	9851	9853	9855	9857	9859	9861	9863	9865	9867	0	1	1	1	2
76	$\bar{1}$·9869	9871	9873	9875	9876	9878	9880	9882	9884	9885					
77	$\bar{1}$·9887	9889	9891	9892	9894	9896	9897	9899	9901	9902					
78	$\bar{1}$·9904	9906	9907	9909	9910	9912	9913	9915	9916	9918					
79	$\bar{1}$·9919	9921	9922	9924	9925	9927	9928	9929	9931	9932					
80	$\bar{1}$·9934	9935	9936	9937	9939	9940	9941	9943	9944	9945					
81	$\bar{1}$·9946	9947	9949	9950	9951	9952	9953	9954	9955	9956					
82	$\bar{1}$·9958	9959	9960	9961	9962	9963	9964	9965	9966	9967					
83	$\bar{1}$·9968	9968	9969	9970	9971	9972	9973	9974	9975	9975			Use Interpolation		
84	$\bar{1}$·9976	9977	9978	9978	9979	9980	9981	9981	9982	9983					
85	$\bar{1}$·9983	9984	9985	9985	9986	9987	9987	9988	9988	9989					
86	$\bar{1}$·9989	9990	9990	9991	9991	9992	9992	9993	9993	9994					
87	$\bar{1}$·9994	9994	9995	9995	9996	9996	9996	9996	9997	9997					
88	$\bar{1}$·9997	9998	9998	9998	9998	9999	9999	9999	9999	9999					
89	$\bar{1}$·9999	9999	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000					

LOG. COSINES

SUBTRACT

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0°	0·0000	0000	0000	0000	0000	0000	0000	0000	0000	$\bar{1}$·9999	Use Interpolation				
1	$\bar{1}$·9999	9999	9999	9999	9999	9999	9998	9998	9998	9998					
2	$\bar{1}$·9997	9997	9997	9997	9996	9996	9996	9995	9995	9994					
3	$\bar{1}$·9994	9994	9993	9993	9992	9992	9991	9991	9990	9990					
4	$\bar{1}$·9989	9989	9988	9988	9987	9987	9986	9985	9985	9984					
5	$\bar{1}$·9983	9983	9982	9981	9981	9980	9979	9978	9978	9977					
6	$\bar{1}$·9976	9975	9975	9974	9973	9972	9971	9970	9969	9968					
7	$\bar{1}$·9968	9967	9966	9965	9964	9963	9962	9961	9960	9959					
8	$\bar{1}$·9958	9956	9955	9954	9953	9952	9951	9950	9949	9947					
9	$\bar{1}$·9946	9945	9944	9943	9941	9940	9939	9937	9936	9935					
10	$\bar{1}$·9934	9932	9931	9929	9928	9927	9925	9924	9922	9921					
11	$\bar{1}$·9919	9918	9916	9915	9913	9912	9910	9909	9907	9906					
12	$\bar{1}$·9904	9902	9901	9899	9897	9896	9894	9892	9891	9889					
13	$\bar{1}$·9887	9885	9884	9882	9880	9878	9876	9875	9873	9871					
14	$\bar{1}$·9869	9867	9865	9863	9861	9859	9857	9855	9853	9851	0	1	1	1	2
15	$\bar{1}$·9849	9847	9845	9843	9841	9839	9837	9835	9833	9831	0	1	1	1	2
16	$\bar{1}$·9828	9826	9824	9822	9820	9817	9815	9813	9811	9808	0	1	1	1	2
17	$\bar{1}$·9806	9804	9801	9799	9797	9794	9792	9789	9787	9785	0	1	1	2	2
18	$\bar{1}$·9782	9780	9777	9775	9772	9770	9767	9764	9762	9759	0	1	1	2	2
19	$\bar{1}$·9757	9754	9751	9749	9746	9743	9741	9738	9735	9733	0	1	1	2	2
20	$\bar{1}$·9730	9727	9724	9722	9719	9716	9713	9710	9707	9704	0	1	1	2	2
21	$\bar{1}$·9702	9699	9696	9693	9690	9687	9684	9681	9678	9675	0	1	1	2	2
22	$\bar{1}$·9672	9669	9666	9662	9659	9656	9653	9650	9647	9643	1	1	2	2	3
23	$\bar{1}$·9640	9637	9634	9631	9627	9624	9621	9617	9614	9611	1	1	2	2	3
24	$\bar{1}$·9607	9604	9601	9597	9594	9590	9587	9583	9580	9576	1	1	2	2	3
25	$\bar{1}$·9573	9569	9566	9562	9558	9555	9551	9548	9544	9540	1	1	2	2	3
26	$\bar{1}$·9537	9533	9529	9525	9522	9518	9514	9510	9506	9503	1	1	2	3	3
27	$\bar{1}$·9499	9495	9491	9487	9483	9479	9475	9471	9467	9463	1	1	2	3	3
28	$\bar{1}$·9459	9455	9451	9447	9443	9439	9435	9431	9427	9422	1	1	2	3	3
29	$\bar{1}$·9418	9414	9410	9406	9401	9397	9393	9388	9384	9380	1	1	2	3	4
30	$\bar{1}$·9375	9371	9367	9362	9358	9353	9349	9344	9340	9335	1	1	2	3	4
31	$\bar{1}$·9331	9326	9322	9317	9312	9308	9303	9298	9294	9289	1	2	2	3	4
32	$\bar{1}$·9284	9279	9275	9270	9265	9260	9255	9251	9246	9241	1	2	2	3	4
33	$\bar{1}$·9236	9231	9226	9221	9216	9211	9206	9201	9196	9191	1	2	3	3	4
34	$\bar{1}$·9186	9181	9175	9170	9165	9160	9155	9149	9144	9139	1	2	3	3	4
35	$\bar{1}$·9134	9128	9123	9118	9112	9107	9101	9096	9091	9085	1	2	3	4	5
36	$\bar{1}$·9080	9074	9069	9063	9057	9052	9046	9041	9035	9029	1	2	3	4	5
37	$\bar{1}$·9023	9018	9012	9006	9000	8995	8989	8983	8977	8971	1	2	3	4	5
38	$\bar{1}$·8965	8959	8953	8947	8941	8935	8929	8923	8917	8911	1	2	3	4	5
39	$\bar{1}$·8905	8899	8893	8887	8880	8874	8868	8862	8855	8849	1	2	3	4	5
40	$\bar{1}$·8843	8836	8830	8823	8817	8810	8804	8797	8791	8784	1	2	3	4	5
41	$\bar{1}$·8778	8771	8765	8758	8751	8745	8738	8731	8724	8718	1	2	3	4	6
42	$\bar{1}$·8711	8704	8697	8690	8683	8676	8669	8662	8655	8648	1	2	3	5	6
43	$\bar{1}$·8641	8634	8627	8620	8613	8606	8598	8591	8584	8577	1	2	4	5	6
44	$\bar{1}$·8569	8562	8555	8547	8540	8532	8525	8517	8510	8502	1	2	4	5	6

SUBTRACT

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	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
45°	$\bar{1}$·8495	8487	8480	8472	8464	8457	8449	8441	8433	8426	1	3	4	5	6
46	$\bar{1}$·8418	8410	8402	8394	8386	8378	8370	8362	8354	8346	1	3	4	5	7
47	$\bar{1}$·8338	8330	8322	8313	8305	8297	8289	8280	8272	8264	1	3	4	6	7
48	$\bar{1}$·8255	8247	8238	8230	8221	8213	8204	8195	8187	8178	1	3	4	6	7
49	$\bar{1}$·8169	8161	8152	8143	8134	8125	8117	8108	8099	8090	1	3	4	6	7
50	$\bar{1}$·8081	8072	8063	8053	8044	8035	8026	8017	8007	7998	2	3	5	6	8
51	$\bar{1}$·7989	7979	7970	7960	7951	7941	7932	7922	7913	7903	2	3	5	6	8
52	$\bar{1}$·7893	7884	7874	7864	7854	7844	7835	7825	7815	7805	2	3	5	7	8
53	$\bar{1}$·7795	7785	7774	7764	7754	7744	7734	7723	7713	7703	2	3	5	7	9
54	$\bar{1}$·7692	7682	7671	7661	7650	7640	7629	7618	7607	7597	2	4	5	7	9
55	$\bar{1}$·7586	7575	7564	7553	7542	7531	7520	7509	7498	7487	2	4	6	7	9
56	$\bar{1}$·7476	7464	7453	7442	7430	7419	7407	7396	7384	7373	2	4	6	8	10
57	$\bar{1}$·7361	7349	7338	7326	7314	7302	7290	7278	7266	7254	2	4	6	8	10
58	$\bar{1}$·7242	7230	7218	7205	7193	7181	7168	7156	7144	7131	2	4	6	8	10
59	$\bar{1}$·7118	7106	7093	7080	7068	7055	7042	7029	7016	7003	2	4	6	9	11
60	$\bar{1}$·6990	6977	6963	6950	6937	6923	6910	6896	6883	6869	2	4	7	9	11
61	$\bar{1}$·6856	6842	6828	6814	6801	6787	6773	6759	6744	6730	2	5	7	9	12
62	$\bar{1}$·6716	6702	6687	6673	6659	6644	6629	6615	6600	6585	2	5	7	10	12
63	$\bar{1}$·6570	6556	6541	6526	6510	6495	6480	6465	6449	6434	3	5	8	10	13
64	$\bar{1}$·6418	6403	6387	6371	6356	6340	6324	6308	6292	6276	3	5	8	11	13
65	$\bar{1}$·6259	6243	6227	6210	6194	6177	6161	6144	6127	6110	3	6	8	11	14
66	$\bar{1}$·6093	6076	6059	6042	6024	6007	5990	5972	5954	5937	3	6	9	12	15
67	$\bar{1}$·5919	5901	5883	5865	5847	5828	5810	5792	5773	5754	3	6	9	12	15
68	$\bar{1}$·5736	5717	5698	5679	5660	5641	5621	5602	5583	5563	3	6	10	13	16
69	$\bar{1}$·5543	5523	5504	5484	5463	5443	5423	5402	5382	5361	3	7	10	14	17
70	$\bar{1}$·5341	5320	5299	5278	5256	5235	5213	5192	5170	5148	4	7	11	14	18
71	$\bar{1}$·5126	5104	5082	5060	5037	5015	4992	4969	4946	4923	4	8	11	15	19
72	$\bar{1}$·4900	4876	4853	4829	4805	4781	4757	4733	4709	4684	4	8	12	16	20
73	$\bar{1}$·4659	4634	4609	4584	4559	4533	4508	4482	4456	4430	4	9	13	17	21
74	$\bar{1}$·4403	4377	4350	4323	4296	4269	4242	4214	4186	4158	5	9	14	18	23
75	$\bar{1}$·4130	4102	4073	4044	4015	3986	3957	3927	3897	3867	5	10	15	20	24
76	$\bar{1}$·3837	3806	3775	3745	3713	3682	3650	3618	3586	3554	5	11	16	21	26
77	$\bar{1}$·3521	3488	3455	3421	3387	3353	3319	3284	3250	3214	6	11	17	23	28
78	$\bar{1}$·3179	3143	3107	3070	3034	2997	2959	2921	2883	2845	6	12	19	25	31
79	$\bar{1}$·2806	2767	2727	2687	2647	2606	2565	2524	2482	2439	7	14	20	27	34
80	$\bar{1}$·2397	2353	2310	2266	2221	2176	2131	2085	2038	1991	8	15	23	30	38
81	$\bar{1}$·1943	1895	1847	1797	1747	1697	1646	1594	1542	1489	8	17	25	34	42
82	$\bar{1}$·1436	1381	1326	1271	1214	1157	1099	1040	0981	0920	10	19	29	38	48
											Difference for 1′				
											1 to 11	13 to 23	25 to 35	37 to 47	49 to 59
83	$\bar{1}$·0859	0797	0734	0670	0605	0539	0472	0403	0334	0264	10	11	11	11	12
84	$\bar{1}$·0192	0120	0046	*9970*	*9894*	*9816*	*9736*	*9655*	*9573*	*9489*	12	13	13	14	14
85	$\bar{2}$·9403	9315	9226	9135	9042	8946	8849	8749	8647	8543	15	15	16	17	18
86	$\bar{2}$·8436	8326	8213	8098	7979	7857	7731	7602	7468	7330	19	20	21	22	23
87	$\bar{2}$·7188	7041	6889	6731	6567	6397	6220	6035	5842	5640	25	27	29	32	35
88	$\bar{2}$·5428	5206	4971	4723	4459	4179	3880	3558	3210	2832					
89	$\bar{2}$·242	$\bar{2}$·196	$\bar{2}$·145	$\bar{2}$·087	$\bar{2}$·020	$\bar{3}$·941	$\bar{3}$·844	$\bar{3}$·719	$\bar{3}$·543	$\bar{3}$·242					

SUBTRACT

Where the integer changes, the numbers are italicised.

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Difference for 1′				
											1 to 11	13 to 23	25 to 35	37 to 47	49 to 59
0°	−∞	$\bar{3}$·242	$\bar{3}$·543	$\bar{3}$·719	$\bar{3}$·844	$\bar{3}$·941	$\bar{2}$·020	$\bar{2}$·087	$\bar{2}$·145	$\bar{2}$·196					
1	$\bar{2}$·2419	2833	3211	3559	3881	4181	4461	4725	4973	5208					
2	$\bar{2}$·5431	5643	5845	6038	6223	6401	6571	6736	6894	7046	35	32	29	27	25
3	$\bar{2}$·7194	7337	7475	7609	7739	7865	7988	8107	8223	8336	23	22	21	20	19
4	$\bar{2}$·8446	8554	8659	8762	8862	8960	9056	9150	9241	9331	18	17	16	15	15
5	$\bar{2}$·9420	9506	9591	9674	9756	9836	9915	9992	*0068*	*0143*	14	14	13	13	12
6	$\bar{1}$·0216	0289	0360	0430	0499	0567	0633	0699	0764	0828	12	12	11	11	11
											1′	2′	3′	4′	5′
7	$\bar{1}$·0891	0954	1015	1076	1135	1194	1252	1310	1367	1423	10	20	29	39	49
8	$\bar{1}$·1478	1533	1587	1640	1693	1745	1797	1848	1898	1948	9	17	26	35	43
9	$\bar{1}$·1997	2046	2094	2142	2189	2236	2282	2328	2374	2419	8	16	23	31	39
10	$\bar{1}$·2463	2507	2551	2594	2637	2680	2722	2764	2805	2846	7	14	21	28	35
11	$\bar{1}$·2887	2927	2967	3006	3046	3085	3123	3162	3200	3237	6	13	19	26	32
12	$\bar{1}$·3275	3312	3349	3385	3422	3458	3493	3529	3564	3599	6	12	18	24	30
13	$\bar{1}$·3634	3668	3702	3736	3770	3804	3837	3870	3903	3935	6	11	17	22	28
14	$\bar{1}$·3968	4000	4032	4064	4095	4127	4158	4189	4220	4250	5	10	16	21	26
15	$\bar{1}$·4281	4311	4341	4371	4400	4430	4459	4488	4517	4546	5	10	15	20	25
16	$\bar{1}$·4575	4603	4632	4660	4688	4716	4744	4771	4799	4826	5	9	14	19	23
17	$\bar{1}$·4853	4880	4907	4934	4961	4987	5014	5040	5066	5092	4	9	13	18	22
18	$\bar{1}$·5118	5143	5169	5195	5220	5245	5270	5295	5320	5345	4	8	13	17	21
19	$\bar{1}$·5370	5394	5419	5443	5467	5491	5516	5539	5563	5587	4	8	12	16	20
20	$\bar{1}$·5611	5634	5658	5681	5704	5727	5750	5773	5796	5819	4	8	12	15	19
21	$\bar{1}$·5842	5864	5887	5909	5932	5954	5976	5998	6020	6042	4	7	11	15	19
22	$\bar{1}$·6064	6086	6108	6129	6151	6172	6194	6215	6236	6257	4	7	11	14	18
23	$\bar{1}$·6279	6300	6321	6341	6362	6383	6404	6424	6445	6465	3	7	10	14	17
24	$\bar{1}$·6486	6506	6527	6547	6567	6587	6607	6627	6647	6667	3	7	10	13	17
25	$\bar{1}$·6687	6706	6726	6746	6765	6785	6804	6824	6843	6863	3	7	10	13	16
26	$\bar{1}$·6882	6901	6920	6939	6958	6977	6996	7015	7034	7053	3	6	9	13	16
27	$\bar{1}$·7072	7090	7109	7128	7146	7165	7183	7202	7220	7238	3	6	9	12	15
28	$\bar{1}$·7257	7275	7293	7311	7330	7348	7366	7384	7402	7420	3	6	9	12	15
29	$\bar{1}$·7438	7455	7473	7491	7509	7526	7544	7562	7579	7597	3	6	9	12	15
30	$\bar{1}$·7614	7632	7649	7667	7684	7701	7719	7736	7753	7771	3	6	9	12	14
31	$\bar{1}$·7788	7805	7822	7839	7856	7873	7890	7907	7924	7941	3	6	9	11	14
32	$\bar{1}$·7958	7975	7992	8008	8025	8042	8059	8075	8092	8109	3	6	8	11	14
33	$\bar{1}$·8125	8142	8158	8175	8191	8208	8224	8241	8257	8274	3	5	8	11	14
34	$\bar{1}$·8290	8306	8323	8339	8355	8371	8388	8404	8420	8436	3	5	8	11	14
35	$\bar{1}$·8452	8468	8484	8501	8517	8533	8549	8565	8581	8597	3	5	8	11	13
36	$\bar{1}$·8613	8629	8644	8660	8676	8692	8708	8724	8740	8755	3	5	8	11	13
37	$\bar{1}$·8771	8787	8803	8818	8834	8850	8865	8881	8897	8912	3	5	8	10	13
38	$\bar{1}$·8928	8944	8959	8975	8990	9006	9022	9037	9053	9068	3	5	8	10	13
39	$\bar{1}$·9084	9099	9115	9130	9146	9161	9176	9192	9207	9223	3	5	8	10	13
40	$\bar{1}$·9238	9254	9269	9284	9300	9315	9330	9346	9361	9376	3	5	8	10	13
41	$\bar{1}$·9392	9407	9422	9438	9453	9468	9483	9499	9514	9529	3	5	8	10	13
42	$\bar{1}$·9544	9560	9575	9590	9605	9621	9636	9651	9666	9681	3	5	8	10	13
43	$\bar{1}$·9697	9712	9727	9742	9757	9772	9788	9803	9818	9833	3	5	8	10	13
44	$\bar{1}$·9848	9864	9879	9894	9909	9924	9939	9955	9970	9985	3	5	8	10	13

Where the integer changes, the numbers are italicised.

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
45°	0·0000	0015	0030	0045	0061	0076	0091	0106	0121	0136	3	5	8	10	13
46	0·0152	0167	0182	0197	0212	0228	0243	0258	0273	0288	3	5	8	10	13
47	0·0303	0319	0334	0349	0364	0379	0395	0410	0425	0440	3	5	8	10	13
48	0·0456	0471	0486	0501	0517	0532	0547	0562	0578	0593	3	5	8	10	13
49	0·0608	0624	0639	0654	0670	0685	0700	0716	0731	0746	3	5	8	10	13
50	0·0762	0777	0793	0808	0824	0839	0854	0870	0885	0901	3	5	8	10	13
51	0·0916	0932	0947	0963	0978	0994	1010	1025	1041	1056	3	5	8	10	13
52	0·1072	1088	1103	1119	1135	1150	1166	1182	1197	1213	3	5	8	10	13
53	0·1229	1245	1260	1276	1292	1308	1324	1340	1356	1371	3	5	8	11	13
54	0·1387	1403	1419	1435	1451	1467	1483	1499	1516	1532	3	5	8	11	13
55	0·1548	1564	1580	1596	1612	1629	1645	1661	1677	1694	3	5	8	11	14
56	0·1710	1726	1743	1759	1776	1792	1809	1825	1842	1858	3	5	8	11	14
57	0·1875	1891	1908	1925	1941	1958	1975	1992	2008	2025	3	6	8	11	14
58	0·2042	2059	2076	2093	2110	2127	2144	2161	2178	2195	3	6	9	11	14
59	0·2212	2229	2247	2264	2281	2299	2316	2333	2351	2368	3	6	9	12	14
60	0·2386	2403	2421	2438	2456	2474	2491	2509	2527	2545	3	6	9	12	15
61	0·2562	2580	2598	2616	2634	2652	2670	2689	2707	2725	3	6	9	12	15
62	0·2743	2762	2780	2798	2817	2835	2854	2872	2891	2910	3	6	9	12	15
63	0·2928	2947	2966	2985	3004	3023	3042	3061	3080	3099	3	6	9	13	16
64	0·3118	3137	3157	3176	3196	3215	3235	3254	3274	3294	3	7	10	13	16
65	0·3313	3333	3353	3373	3393	3413	3433	3453	3473	3494	3	7	10	13	17
66	0·3514	3535	3555	3576	3596	3617	3638	3659	3679	3700	3	7	10	14	17
67	0·3721	3743	3764	3785	3806	3828	3849	3871	3892	3914	4	7	11	14	18
68	0·3936	3958	3980	4002	4024	4046	4068	4091	4113	4136	4	7	11	15	19
69	0·4158	4181	4204	4227	4250	4273	4296	4319	4342	4366	4	8	12	15	19
70	0·4389	4413	4437	4461	4484	4509	4533	4557	4581	4606	4	8	12	16	20
71	0·4630	4655	4680	4705	4730	4755	4780	4805	4831	4857	4	8	13	17	21
72	0·4882	4908	4934	4960	4986	5013	5039	5066	5093	5120	4	9	13	18	22
73	0·5147	5174	5201	5229	5256	5284	5312	5340	5368	5397	5	9	14	19	23
74	0·5425	5454	5483	5512	5541	5570	5600	5629	5659	5689	5	10	15	20	25
75	0·5719	5750	5780	5811	5842	5873	5905	5936	5968	6000	5	10	16	21	26
76	0·6032	6065	6097	6130	6163	6196	6230	6264	6298	6332	6	11	17	22	28
77	0·6366	6401	6436	6471	6507	6542	6578	6615	6651	6688	6	12	18	24	30
78	0·6725	6763	6800	6838	6877	6915	6954	6994	7033	7073	6	13	19	26	32
79	0·7113	7154	7195	7236	7278	7320	7363	7406	7449	7493	7	14	21	28	35
80	0·7537	7581	7626	7672	7718	7764	7811	7858	7906	7954	8	16	23	31	39
81	0·8003	8052	8102	8152	8203	8255	8307	8360	8413	8467	9	17	26	35	43
82	0·8522	8577	8633	8690	8748	8806	8865	8924	8985	9046	10	20	29	39	49
											Difference for 1′				
											1 to 11	13 to 23	25 to 35	37 to 47	49 to 59
83	0·9109	9172	9236	9301	9367	9433	9501	9570	9640	9711	11	11	11	12	12
84	0·9784	9857	9932	*0008*	*0085*	*0164*	*0244*	*0326*	*0409*	*0494*	12	13	13	14	14
85	1·0580	0669	0759	0850	0944	1040	1138	1238	1341	1446	15	15	16	17	18
86	1·1554	1664	1777	1893	2012	2135	2261	2391	2525	2663	19	20	21	22	23
87	1·2806	2954	3106	3264	3429	3599	3777	3962	4155	4357	25	27	29	32	35
88	1·4569	4792	5027	5275	5539	5819	6119	6441	6789	7167					
89	1·758	1·804	1·855	1·913	1·980	2·059	2·156	2·281	2·457	2·758					

Where the integer changes, the numbers are italicised.

LOG. COTANGENTS

SUBTRACT

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Difference for 1′				
											1 to 11	13 to 23	25 to 35	37 to 47	49 to 59
0°	+∞	2·758	2·457	2·281	2·156	2·059	1·980	1·913	1·855	1·804					
1	1·7581	7167	6789	6441	6119	5819	5539	5275	5027	4792					
2	1·4569	4357	4155	3962	3777	3599	3429	3264	3106	2954	35	32	29	27	25
3	1·2806	2663	2525	2391	2261	2135	2012	1893	1777	1664	23	22	21	20	19
4	1·1554	1446	1341	1238	1138	1040	0944	0850	0759	0669	18	17	16	15	15
5	1·0580	0494	0409	0326	0244	0164	0085	0008	*9932*	*9857*	14	14	13	13	12
6	0·9784	9711	9640	9570	9501	9433	9367	9301	9236	9172	12	12	11	11	11
											1′	2′	3′	4′	5′
7	0·9109	9046	8985	8924	8865	8806	8748	8690	8633	8577	10	20	29	39	49
8	0·8522	8467	8413	8360	8307	8255	8203	8152	8102	8052	9	17	26	35	43
9	0·8003	7954	7906	7858	7811	7764	7718	7672	7626	7581	8	16	23	31	39
10	0·7537	7493	7449	7406	7363	7320	7278	7236	7195	7154	7	14	21	28	35
11	0·7113	7073	7033	6994	6954	6915	6877	6838	6800	6763	6	13	19	26	32
12	0·6725	6688	6651	6615	6578	6542	6507	6471	6436	6401	6	12	18	24	30
13	0·6366	6332	6298	6264	6230	6196	6163	6130	6097	6065	6	11	17	22	28
14	0·6032	6000	5968	5936	5905	5873	5842	5811	5780	5750	5	10	16	21	26
15	0·5719	5689	5659	5629	5600	5570	5541	5512	5483	5454	5	10	15	20	25
16	0·5425	5397	5368	5340	5312	5284	5256	5229	5201	5174	5	9	14	19	23
17	0·5147	5120	5093	5066	5039	5013	4986	4960	4934	4908	4	9	13	18	22
18	0·4882	4857	4831	4805	4780	4755	4730	4705	4680	4655	4	8	13	17	21
19	0·4630	4606	4581	4557	4533	4509	4484	4461	4437	4413	4	8	12	16	20
20	0·4389	4366	4342	4319	4296	4273	4250	4227	4204	4181	4	8	12	15	19
21	0·4158	4136	4113	4091	4068	4046	4024	4002	3980	3958	4	7	11	15	19
22	0·3936	3914	3892	3871	3849	3828	3806	3785	3764	3743	4	7	11	14	18
23	0·3721	3700	3679	3659	3638	3617	3596	3576	3555	3535	3	7	10	14	17
24	0·3514	3494	3473	3453	3433	3413	3393	3373	3353	3333	3	7	10	13	17
25	0·3313	3294	3274	3254	3235	3215	3196	3176	3157	3137	3	7	10	13	16
26	0·3118	3099	3080	3061	3042	3023	3004	2985	2966	2947	3	6	9	13	16
27	0·2928	2910	2891	2872	2854	2835	2817	2798	2780	2762	3	6	9	12	15
28	0·2743	2725	2707	2689	2670	2652	2634	2616	2598	2580	3	6	9	12	15
29	0·2562	2545	2527	2509	2491	2474	2456	2438	2421	2403	3	6	9	12	15
30	0·2386	2368	2351	2333	2316	2299	2281	2264	2247	2229	3	6	9	12	14
31	0·2212	2195	2178	2161	2144	2127	2110	2093	2076	2059	3	6	9	11	14
32	0·2042	2025	2008	1992	1975	1958	1941	1925	1908	1891	3	6	8	11	14
33	0·1875	1858	1842	1825	1809	1792	1776	1759	1743	1726	3	5	8	11	14
34	0·1710	1694	1677	1661	1645	1629	1612	1596	1580	1564	3	5	8	11	14
35	0·1548	1532	1516	1499	1483	1467	1451	1435	1419	1403	3	5	8	11	13
36	0·1387	1371	1356	1340	1324	1308	1292	1276	1260	1245	3	5	8	11	13
37	0·1229	1213	1197	1182	1166	1150	1135	1119	1103	1088	3	5	8	10	13
38	0·1072	1056	1041	1025	1010	0994	0978	0963	0947	0932	3	5	8	10	13
39	0·0916	0901	0885	0870	0854	0839	0824	0808	0793	0777	3	5	8	10	13
40	0·0762	0746	0731	0716	0700	0685	0670	0654	0639	0624	3	5	8	10	13
41	0·0608	0593	0578	0562	0547	0532	0517	0501	0486	0471	3	5	8	10	13
42	0·0456	0440	0425	0410	0395	0379	0364	0349	0334	0319	3	5	8	10	13
43	0·0303	0288	0273	0258	0243	0228	0212	0197	0182	0167	3	5	8	10	13
44	0·0152	0136	0121	0106	0091	0076	0061	0045	0030	0015	3	5	8	10	13

SUBTRACT

Where the integer changes, the numbers are italicised.

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
											SUBTRACT				
45°	0·0000	*9985*	*9970*	*9955*	*9939*	*9924*	*9909*	*9894*	*9879*	*9864*	3	5	8	10	13
46	$\bar{1}$·9848	9833	9818	9803	9788	9772	9757	9742	9727	9712	3	5	8	10	13
47	$\bar{1}$·9697	9681	9666	9651	9636	9621	9605	9590	9575	9560	3	5	8	10	13
48	$\bar{1}$·9544	9529	9514	9499	9483	9468	9453	9438	9422	9407	3	5	8	10	13
49	$\bar{1}$·9392	9376	9361	9346	9330	9315	9300	9284	9269	9254	3	5	8	10	13
50	$\bar{1}$·9238	9223	9207	9192	9176	9161	9146	9130	9115	9099	3	5	8	10	13
51	$\bar{1}$·9084	9068	9053	9037	9022	9006	8990	8975	8959	8944	3	5	8	10	13
52	$\bar{1}$·8928	8912	8897	8881	8865	8850	8834	8818	8803	8787	3	5	8	10	13
53	$\bar{1}$·8771	8755	8740	8724	8708	8692	8676	8660	8644	8629	3	5	8	11	13
54	$\bar{1}$·8613	8597	8581	8565	8549	8533	8517	8501	8484	8468	3	5	8	11	13
55	$\bar{1}$·8452	8436	8420	8404	8388	8371	8355	8339	8323	8306	3	5	8	11	14
56	$\bar{1}$·8290	8274	8257	8241	8224	8208	8191	8175	8158	8142	3	5	8	11	14
57	$\bar{1}$·8125	8109	8092	8075	8059	8042	8025	8008	7992	7975	3	6	8	11	14
58	$\bar{1}$·7958	7941	7924	7907	7890	7873	7856	7839	7822	7805	3	6	9	11	14
59	$\bar{1}$·7788	7771	7753	7736	7719	7701	7684	7667	7649	7632	3	6	9	12	14
60	$\bar{1}$·7614	7597	7579	7562	7544	7526	7509	7491	7473	7455	3	6	9	12	15
61	$\bar{1}$·7438	7420	7402	7384	7366	7348	7330	7311	7293	7275	3	6	9	12	15
62	$\bar{1}$·7257	7238	7220	7202	7183	7165	7146	7128	7109	7090	3	6	9	12	15
63	$\bar{1}$·7072	7053	7034	7015	6996	6977	6958	6939	6920	6901	3	6	9	13	16
64	$\bar{1}$·6882	6863	6843	6824	6804	6785	6765	6746	6726	6706	3	7	10	13	16
65	$\bar{1}$·6687	6667	6647	6627	6607	6587	6567	6547	6527	6506	3	7	10	13	17
66	$\bar{1}$·6486	6465	6445	6424	6404	6383	6362	6341	6321	6300	3	7	10	14	17
67	$\bar{1}$·6279	6257	6236	6215	6194	6172	6151	6129	6108	6086	4	7	11	14	18
68	$\bar{1}$·6064	6042	6020	5998	5976	5954	5932	5909	5887	5864	4	7	11	15	19
69	$\bar{1}$·5842	5819	5796	5773	5750	5727	5704	5681	5658	5634	4	8	12	15	19
70	$\bar{1}$·5611	5587	5563	5539	5516	5491	5467	5443	5419	5394	4	8	12	16	20
71	$\bar{1}$·5370	5345	5320	5295	5270	5245	5220	5195	5169	5143	4	8	13	17	21
72	$\bar{1}$·5118	5092	5066	5040	5014	4987	4961	4934	4907	4880	4	9	13	18	22
73	$\bar{1}$·4853	4826	4799	4771	4744	4716	4688	4660	4632	4603	5	9	14	19	23
74	$\bar{1}$·4575	4546	4517	4488	4459	4430	4400	4371	4341	4311	5	10	15	20	25
75	$\bar{1}$·4281	4250	4220	4189	4158	4127	4095	4064	4032	4000	5	10	16	21	26
76	$\bar{1}$·3968	3935	3903	3870	3837	3804	3770	3736	3702	3668	6	11	17	22	28
77	$\bar{1}$·3634	3599	3564	3529	3493	3458	3422	3385	3349	3312	6	12	18	24	30
78	$\bar{1}$·3275	3237	3200	3162	3123	3085	3046	3006	2967	2927	6	13	19	26	32
79	$\bar{1}$·2887	2846	2805	2764	2722	2680	2637	2594	2551	2507	7	14	21	28	35
80	$\bar{1}$·2463	2419	2374	2328	2282	2236	2189	2142	2094	2046	8	16	23	31	39
81	$\bar{1}$·1997	1948	1898	1848	1797	1745	1693	1640	1587	1533	9	17	26	35	43
82	$\bar{1}$·1478	1423	1367	1310	1252	1194	1135	1076	1015	0954	10	20	29	39	49
											Difference for 1′				
											1 to 11	13 to 23	25 to 35	37 to 47	49 to 59
83	$\bar{1}$·0891	0828	0764	0699	0633	0567	0499	0430	0360	0289	11	11	11	12	12
84	$\bar{1}$·0216	0143	0068	*9992*	*9915*	*9836*	*9756*	*9674*	*9591*	*9506*	12	13	13	14	14
85	$\bar{2}$·9420	9331	9241	9150	9056	8960	8862	8762	8659	8554	15	15	16	17	18
86	$\bar{2}$·8446	8336	8223	8107	7988	7865	7739	7609	7475	7337	19	20	21	22	23
87	$\bar{2}$·7194	7046	6894	6736	6571	6401	6223	6038	5845	5643	25	27	29	32	35
88	$\bar{2}$·5431	5208	4973	4725	4461	4181	3881	3559	3211	2833					
89	$\bar{2}$·242	$\bar{2}$·196	$\bar{2}$·145	$\bar{2}$·087	$\bar{2}$·020	$\bar{3}$·941	$\bar{3}$·844	$\bar{3}$·719	$\bar{3}$·543	$\bar{3}$·242					
											SUBTRACT				

Where the integer changes, the numbers are italicised.

LOG. COSECANTS

SUBTRACT

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Difference for 1′				
											1 to 11	13 to 23	25 to 35	37 to 47	49 to 59
0°	+∞	2·758	2·457	2·281	2·156	2·059	1·980	1·913	1·855	1·804					
1	1·7581	7168	6790	6442	6120	5821	5541	5277	5029	4794	—	—	—	—	—
2	1·4572	4360	4158	3965	3780	3603	3433	3269	3111	2959	35	32	29	27	25
3	1·2812	2670	2532	2398	2269	2143	2021	1902	1787	1674	23	22	21	20	19
4	1·1564	1457	1353	1251	1151	1054	0958	0865	0774	0685	18	17	16	15	15
5	1·0597	0511	0427	0345	0264	0184	0106	0030	*9954*	*9880*	14	14	13	13	12
6	0·9808	9736	9666	9597	9528	9461	9395	9330	9266	9203	12	11	11	11	10
7	0·9141	9080	9019	8960	8901	8843	8786	8729	8674	8619	10	10	10	9	9
											1′	2′	3′	4′	5′
8	0·8564	8511	8458	8406	8354	8303	8253	8203	8153	8105	8	17	25	34	42
9	0·8057	8009	7962	7915	7869	7824	7779	7734	7690	7647	8	15	23	30	38
10	0·7603	7561	7518	7476	7435	7394	7353	7313	7273	7233	7	14	20	27	34
11	0·7194	7155	7117	7079	7041	7003	6966	6930	6893	6857	6	12	19	25	31
12	0·6821	6786	6750	6716	6681	6647	6613	6579	6545	6512	6	11	17	23	28
13	0·6479	6446	6414	6382	6350	6318	6287	6255	6225	6194	5	11	16	21	26
14	0·6163	6133	6103	6073	6043	6014	5985	5956	5927	5898	5	10	15	20	24
15	0·5870	5842	5814	5786	5758	5731	5704	5677	5650	5623	5	9	14	18	23
16	0·5597	5570	5544	5518	5492	5467	5441	5416	5391	5366	4	9	13	17	21
17	0·5341	5316	5291	5267	5243	5219	5195	5171	5147	5124	4	8	12	16	20
18	0·5100	5077	5054	5031	5008	4985	4963	4940	4918	4896	4	8	11	15	19
19	0·4874	4852	4830	4808	4787	4765	4744	4722	4701	4680	4	7	11	14	18
20	0·4659	4639	4618	4598	4577	4557	4537	4516	4496	4477	3	7	10	14	17
21	0·4457	4437	4417	4398	4379	4359	4340	4321	4302	4283	3	6	10	13	16
22	0·4264	4246	4227	4208	4190	4172	4153	4135	4117	4099	3	6	9	12	15
23	0·4081	4063	4046	4028	4010	3993	3976	3958	3941	3924	3	6	9	12	15
24	0·3907	3890	3873	3856	3839	3823	3806	3790	3773	3757	3	6	8	11	14
25	0·3741	3724	3708	3692	3676	3660	3644	3629	3613	3597	3	5	8	11	13
26	0·3582	3566	3551	3535	3520	3505	3490	3474	3459	3444	3	5	8	10	13
27	0·3430	3415	3400	3385	3371	3356	3341	3327	3313	3298	2	5	7	10	12
28	0·3284	3270	3256	3241	3227	3213	3199	3186	3172	3158	2	5	7	9	12
29	0·3144	3131	3117	3104	3090	3077	3063	3050	3037	3023	2	4	7	9	11
30	0·3010	2997	2984	2971	2958	2945	2932	2920	2907	2894	2	4	6	9	11
31	0·2882	2869	2856	2844	2832	2819	2807	2795	2782	2770	2	4	6	8	10
32	0·2758	2746	2734	2722	2710	2698	2686	2674	2662	2651	2	4	6	8	10
33	0·2639	2627	2616	2604	2593	2581	2570	2558	2547	2536	2	4	6	8	10
34	0·2524	2513	2502	2491	2480	2469	2458	2447	2436	2425	2	4	6	7	9
35	0·2414	2403	2393	2382	2371	2360	2350	2339	2329	2318	2	4	5	7	9
36	0·2308	2297	2287	2277	2266	2256	2246	2236	2226	2215	2	3	5	7	9
37	0·2205	2195	2185	2175	2165	2156	2146	2136	2126	2116	2	3	5	7	8
38	0·2107	2097	2087	2078	2068	2059	2049	2040	2030	2021	2	3	5	6	8
39	0·2011	2002	1993	1983	1974	1965	1956	1947	1937	1928	2	3	5	6	8
40	0·1919	1910	1901	1892	1883	1875	1866	1857	1848	1839	1	3	4	6	7
41	0·1831	1822	1813	1805	1796	1787	1779	1770	1762	1753	1	3	4	6	7
42	0·1745	1736	1728	1720	1711	1703	1695	1687	1678	1670	1	3	4	6	7
43	0·1662	1654	1646	1638	1630	1622	1614	1606	1598	1590	1	3	4	5	7
44	0·1582	1574	1567	1559	1551	1543	1536	1528	1520	1513	1	3	4	5	6

SUBTRACT

Where the integer changes, the numbers are italicised.

											SUBTRACT				
	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
45°	0·1505	1498	1490	1483	1475	1468	1460	1453	1445	1438	1	2	4	5	6
46	0·1431	1423	1416	1409	1402	1394	1387	1380	1373	1366	1	2	4	5	6
47	0·1359	1352	1345	1338	1331	1324	1317	1310	1303	1296	1	2	3	5	6
48	0·1289	1282	1276	1269	1262	1255	1249	1242	1235	1229	1	2	3	4	6
49	0·1222	1216	1209	1203	1196	1190	1183	1177	1170	1164	1	2	3	4	5
50	0·1157	1151	1145	1138	1132	1126	1120	1113	1107	1101	1	2	3	4	5
51	0·1095	1089	1083	1077	1071	1065	1059	1053	1047	1041	1	2	3	4	5
52	0·1035	1029	1023	1017	1011	1005	1000	0994	0988	0982	1	2	3	4	5
53	0·0977	0971	0965	0959	0954	0948	0943	0937	0931	0926	1	2	3	4	5
54	0·0920	0915	0909	0904	0899	0893	0888	0882	0877	0872	1	2	3	4	5
55	0·0866	0861	0856	0851	0845	0840	0835	0830	0825	0819	1	2	3	3	4
56	0·0814	0809	0804	0799	0794	0789	0784	0779	0774	0769	1	2	3	3	4
57	0·0764	0759	0754	0749	0745	0740	0735	0730	0725	0721	1	2	2	3	4
58	0·0716	0711	0706	0702	0697	0692	0688	0683	0678	0674	1	2	2	3	4
59	0·0669	0665	0660	0656	0651	0647	0642	0638	0633	0629	1	1	2	3	4
60	0·0625	0620	0616	0612	0607	0603	0599	0594	0590	0586	1	1	2	3	4
61	0·0582	0578	0573	0569	0565	0561	0557	0553	0549	0545	1	1	2	3	3
62	0·0541	0537	0533	0529	0525	0521	0517	0513	0509	0505	1	1	2	3	3
63	0·0501	0497	0494	0490	0486	0482	0478	0475	0471	0467	1	1	2	3	3
64	0·0463	0460	0456	0452	0449	0445	0442	0438	0434	0431	1	1	2	2	3
65	0·0427	0424	0420	0417	0413	0410	0406	0403	0399	0396	1	1	2	2	3
66	0·0393	0389	0386	0383	0379	0376	0373	0369	0366	0363	1	1	2	2	3
67	0·0360	0357	0353	0350	0347	0344	0341	0338	0334	0331	1	1	2	2	3
68	0·0328	0325	0322	0319	0316	0313	0310	0307	0304	0301	0	1	1	2	2
69	0·0298	0296	0293	0290	0287	0284	0281	0278	0276	0273	0	1	1	2	2
70	0·0270	0267	0265	0262	0259	0257	0254	0251	0249	0246	0	1	1	2	2
71	0·0243	0241	0238	0236	0233	0230	0228	0225	0223	0220	0	1	1	2	2
72	0·0218	0215	0213	0211	0208	0206	0203	0201	0199	0196	0	1	1	2	2
73	0·0194	0192	0189	0187	0185	0183	0180	0178	0176	0174	0	1	1	1	2
74	0·0172	0169	0167	0165	0163	0161	0159	0157	0155	0153	0	1	1	1	2
75	0·0151	0149	0147	0145	0143	0141	0139	0137	0135	0133	0	1	1	1	2
76	0·0131	0129	0127	0125	0124	0122	0120	0118	0116	0115					
77	0·0113	0111	0109	0108	0106	0104	0103	0101	0099	0098					
78	0·0096	0094	0093	0091	0090	0088	0087	0085	0084	0082					
79	0·0084	0079	0078	0076	0075	0073	0072	0071	0069	0068					
80	0·0066	0065	0064	0063	0061	0060	0059	0057	0056	0055					
81	0·0054	0053	0051	0050	0049	0048	0047	0046	0045	0044					
82	0·0042	0041	0040	0039	0038	0037	0036	0035	0034	0033					
83	0·0032	0032	0031	0030	0029	0028	0027	0026	0025	0025	Use Interpolation				
84	0·0024	0023	0022	0022	0021	0020	0019	0019	0018	0017					
85	0·0017	0016	0015	0015	0014	0013	0013	0012	0012	0011					
86	0·0011	0010	0010	0009	0009	0008	0008	0007	0007	0006					
87	0·0006	0006	0005	0005	0004	0004	0004	0004	0003	0003					
88	0·0003	0002	0002	0002	0002	0001	0001	0001	0001	0001					
89	0·0001	0001	0000	0000	0000	0000	0000	0000	0000	0000					

SUBTRACT

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0°	0·0000	0000	0000	0000	0000	0000	0000	0000	0000	0001	Use Interpolation.				
1	0·0001	0001	0001	0001	0001	0001	0002	0002	0002	0002					
2	0·0003	0003	0003	0004	0004	0004	0004	0005	0005	0006					
3	0·0006	0006	0007	0007	0008	0008	0009	0009	0010	0010					
4	0·0011	0011	0012	0012	0013	0013	0014	0015	0015	0016					
5	0·0017	0017	0018	0019	0019	0020	0021	0022	0022	0023					
6	0·0024	0025	0025	0026	0027	0028	0029	0030	0031	0032					
7	0·0032	0033	0034	0035	0036	0037	0038	0039	0040	0041					
8	0·0042	0044	0045	0046	0047	0048	0049	0050	0051	0053					
9	0·0054	0055	0056	0057	0059	0060	0061	0063	0064	0065					
10	0·0066	0068	0069	0071	0072	0073	0075	0076	0078	0079					
11	0·0081	0082	0084	0085	0087	0088	0090	0091	0093	0094					
12	0·0096	0098	0099	0101	0103	0104	0106	0108	0109	0111					
13	0·0113	0115	0116	0118	0120	0122	0124	0125	0127	0129					
14	0·0131	0133	0135	0137	0139	0141	0143	0145	0147	0149	0	1	1	1	2
15	0·0151	0153	0155	0157	0159	0161	0163	0165	0167	0169	0	1	1	1	2
16	0·0172	0174	0176	0178	0180	0183	0185	0187	0189	0192	0	1	1	1	2
17	0·0194	0196	0199	0201	0203	0206	0208	0211	0213	0215	0	1	1	2	2
18	0·0218	0220	0223	0225	0228	0230	0233	0236	0238	0241	0	1	1	2	2
19	0·0243	0246	0249	0251	0254	0257	0259	0262	0265	0267	0	1	1	2	2
20	0·0270	0273	0276	0278	0281	0284	0287	0290	0293	0296	0	1	1	2	2
21	0·0298	0301	0304	0307	0310	0313	0316	0319	0322	0325	0	1	1	2	2
22	0·0328	0331	0334	0338	0341	0344	0347	0350	0353	0357	1	1	2	2	3
23	0·0360	0363	0366	0369	0373	0376	0379	0383	0386	0389	1	1	2	2	3
24	0·0393	0396	0399	0403	0406	0410	0413	0417	0420	0424	1	1	2	2	3
25	0·0427	0431	0434	0438	0442	0445	0449	0452	0456	0460	1	1	2	2	3
26	0·0463	0467	0471	0475	0478	0482	0486	0490	0494	0497	1	1	2	3	3
27	0·0501	0505	0509	0513	0517	0521	0525	0529	0533	0537	1	1	2	3	3
28	0·0541	0545	0549	0553	0557	0561	0565	0569	0573	0578	1	1	2	3	3
29	0·0582	0586	0590	0594	0599	0603	0607	0612	0616	0620	1	1	2	3	4
30	0·0625	0629	0633	0638	0642	0647	0651	0656	0660	0665	1	1	2	3	4
31	0·0669	0674	0678	0683	0688	0692	0697	0702	0706	0711	1	2	2	3	4
32	0·0716	0721	0725	0730	0735	0740	0745	0749	0754	0759	1	2	2	3	4
33	0·0764	0769	0774	0779	0784	0789	0794	0799	0804	0809	1	2	3	3	4
34	0·0814	0819	0825	0830	0835	0840	0845	0851	0856	0861	1	2	3	3	4
35	0·0866	0872	0877	0882	0888	0893	0899	0904	0909	0915	1	2	3	4	5
36	0·0920	0926	0931	0937	0943	0948	0954	0959	0965	0971	1	2	3	4	5
37	0·0977	0982	0988	0994	1000	1005	1011	1017	1023	1029	1	2	3	4	5
38	0·1035	1041	1047	1053	1059	1065	1071	1077	1083	1089	1	2	3	4	5
39	0·1095	1101	1107	1113	1120	1126	1132	1138	1145	1151	1	2	3	4	5
40	0·1157	1164	1170	1177	1183	1190	1196	1203	1209	1216	1	2	3	4	5
41	0·1222	1229	1235	1242	1249	1255	1262	1269	1276	1282	1	2	3	4	6
42	0·1289	1296	1303	1310	1317	1324	1331	1338	1345	1352	1	2	3	5	6
43	0·1359	1366	1373	1380	1387	1394	1402	1409	1416	1423	1	2	4	5	6
44	0·1431	1438	1445	1453	1460	1468	1475	1483	1490	1498	1	2	4	5	6

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
45°	0·1505	1513	1520	1528	1536	1543	1551	1559	1567	1574	1	3	4	5	6
46	0·1582	1590	1598	1606	1614	1622	1630	1638	1646	1654	1	3	4	5	7
47	0·1662	1670	1678	1687	1695	1703	1711	1720	1728	1736	1	3	4	6	7
48	0·1745	1753	1762	1770	1779	1787	1796	1805	1813	1822	1	3	4	6	7
49	0·1831	1839	1848	1857	1866	1875	1883	1892	1901	1910	1	3	4	6	7
50	0·1919	1928	1937	1947	1956	1965	1974	1983	1993	2002	2	3	5	6	8
51	0·2011	2021	2030	2040	2049	2059	2068	2078	2087	2097	2	3	5	6	8
52	0·2107	2116	2126	2136	2146	2156	2165	2175	2185	2195	2	3	5	7	8
53	0·2205	2215	2226	2236	2246	2256	2266	2277	2287	2297	2	3	5	7	9
54	0·2308	2318	2329	2339	2350	2360	2371	2382	2393	2403	2	4	5	7	9
55	0·2414	2425	2436	2447	2458	2469	2480	2491	2502	2513	2	4	6	7	9
56	0·2524	2536	2547	2558	2570	2581	2593	2604	2616	2627	2	4	6	8	10
57	0·2639	2651	2662	2674	2686	2698	2710	2722	2734	2746	2	4	6	8	10
58	0·2758	2770	2782	2795	2807	2819	2832	2844	2856	2869	2	4	6	8	10
59	0·2882	2894	2907	2920	2932	2945	2958	2971	2984	2997	2	4	6	9	11
60	0·3010	3023	3037	3050	3063	3077	3090	3104	3117	3131	2	4	7	9	11
61	0·3144	3158	3172	3186	3199	3213	3227	3241	3256	3270	2	5	7	9	12
62	0·3284	3298	3313	3327	3341	3356	3371	3385	3400	3415	2	5	7	10	12
63	0·3430	3444	3459	3474	3490	3505	3520	3535	3551	3566	3	5	8	10	13
64	0·3582	3597	3613	3629	3644	3660	3676	3692	3708	3724	3	5	8	11	13
65	0·3741	3757	3773	3790	3806	3823	3839	3856	3873	3890	3	6	8	11	14
66	0·3907	3924	3941	3958	3976	3993	4010	4028	4046	4063	3	6	9	12	15
67	0·4081	4099	4117	4135	4153	4172	4190	4208	4227	4246	3	6	9	12	15
68	0·4264	4283	4302	4321	4340	4359	4379	4398	4417	4437	3	6	10	13	16
69	0·4457	4477	4496	4516	4537	4557	4577	4598	4618	4639	3	7	10	14	17
70	0·4659	4680	4701	4722	4744	4765	4787	4808	4830	4852	4	7	11	14	18
71	0·4874	4896	4918	4940	4963	4985	5008	5031	5054	5077	4	8	11	15	19
72	0·5100	5124	5147	5171	5195	5219	5243	5267	5291	5316	4	8	12	16	20
73	0·5341	5366	5391	5416	5441	5467	5492	5518	5544	5570	4	9	13	17	21
74	0·5597	5623	5650	5677	5704	5731	5758	5786	5814	5842	5	9	14	18	23
75	0·5870	5898	5927	5956	5985	6014	6043	6073	6103	6133	5	10	15	20	24
76	0·6163	6194	6225	6255	6287	6318	6350	6382	6414	6446	5	11	16	21	26
77	0·6479	6512	6545	6579	6613	6647	6681	6716	6750	6786	6	11	17	23	28
78	0·6821	6857	6893	6930	6966	7003	7041	7079	7117	7155	6	12	19	25	31
79	0·7194	7233	7273	7313	7353	7394	7435	7476	7518	7561	7	14	20	27	34
80	0·7603	7647	7690	7734	7779	7824	7869	7915	7962	8009	8	15	23	30	38
81	0·8057	8105	8153	8203	8253	8303	8354	8406	8458	8511	8	17	25	34	42
											Difference for 1′				
											1 to 11	13 to 23	25 to 35	37 to 47	49 to 59
82	0·8564	8619	8674	8729	8786	8843	8901	8960	9019	9080	9	9	10	10	10
83	0·9141	9203	9266	9330	9395	9461	9528	9597	9666	9736	10	11	11	11	12
84	0·9808	9880	9954	*0030*	*0106*	*0184*	*0264*	*0345*	*0427*	*0511*	12	13	13	14	14
85	1·0597	0685	0774	0865	0958	1054	1151	1251	1353	1457	15	15	16	17	18
86	1·1564	1674	1787	1902	2021	2143	2269	2398	2532	2670	19	20	21	22	23
87	1·2812	2959	3111	3269	3433	3603	3780	3965	4158	4360	25	27	29	32	35
88	1·4572	4794	5029	5277	5541	5821	6120	6442	6790	7168					
89	1·7581	1·804	1·855	1·913	1·980	2·059	2·156	2·281	2·457	2·758					

Where the integer changes, the numbers are italicised.

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0°	·0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	15
1	·0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	·0349	0366	0384	0401	0419	0436	0454	0471	0488	0506	3	6	9	12	15
3	·0523	0541	0558	0576	0593	0610	0628	0645	0663	0680	3	6	9	12	15
4	·0698	0715	0732	0750	0767	0785	0802	0819	0837	0854	3	6	9	12	14
5	·0872	0889	0906	0924	0941	0958	0976	0993	1011	1028	3	6	9	12	14
6	·1045	1063	1080	1097	1115	1132	1149	1167	1184	1201	3	6	9	12	14
7	·1219	1236	1253	1271	1288	1305	1323	1340	1357	1374	3	6	9	12	14
8	·1392	1409	1426	1444	1461	1478	1495	1513	1530	1547	3	6	9	12	14
9	·1564	1582	1599	1616	1633	1650	1668	1685	1702	1719	3	6	9	11	14
10	·1736	1754	1771	1788	1805	1822	1840	1857	1874	1891	3	6	9	11	14
11	·1908	1925	1942	1959	1977	1994	2011	2028	2045	2062	3	6	9	11	14
12	·2079	2096	2113	2130	2147	2164	2181	2198	2215	2233	3	6	9	11	14
13	·2250	2267	2284	2300	2317	2334	2351	2368	2385	2402	3	6	8	11	14
14	·2419	2436	2453	2470	2487	2504	2521	2538	2554	2571	3	6	8	11	14
15	·2588	2605	2622	2639	2656	2672	2689	2706	2723	2740	3	6	8	11	14
16	·2756	2773	2790	2807	2823	2840	2857	2874	2890	2907	3	6	8	11	14
17	·2924	2940	2957	2974	2990	3007	3024	3040	3057	3074	3	6	8	11	14
18	·3090	3107	3123	3140	3156	3173	3190	3206	3223	3239	3	6	8	11	14
19	·3256	3272	3289	3305	3322	3338	3355	3371	3387	3404	3	5	8	11	14
20	·3420	3437	3453	3469	3486	3502	3518	3535	3551	3567	3	5	8	11	14
21	·3584	3600	3616	3633	3649	3665	3681	3697	3714	3730	3	5	8	11	14
22	·3746	3762	3778	3795	3811	3827	3843	3859	3875	3891	3	5	8	11	13
23	·3907	3923	3939	3955	3971	3987	4003	4019	4035	4051	3	5	8	11	13
24	·4067	4083	4099	4115	4131	4147	4163	4179	4195	4210	3	5	8	11	13
25	·4226	4242	4258	4274	4289	4305	4321	4337	4352	4368	3	5	8	11	13
26	·4384	4399	4415	4431	4446	4462	4478	4493	4509	4524	3	5	8	10	13
27	·4540	4555	4571	4586	4602	4617	4633	4648	4664	4679	3	5	8	10	13
28	·4695	4710	4726	4741	4756	4772	4787	4802	4818	4833	3	5	8	10	13
29	·4848	4863	4879	4894	4909	4924	4939	4955	4970	4985	3	5	8	10	13
30	·5000	5015	5030	5045	5060	5075	5090	5105	5120	5135	3	5	8	10	13
31	·5150	5165	5180	5195	5210	5225	5240	5255	5270	5284	2	5	7	10	12
32	·5299	5314	5329	5344	5358	5373	5388	5402	5417	5432	2	5	7	10	12
33	·5446	5461	5476	5490	5505	5519	5534	5548	5563	5577	2	5	7	10	12
34	·5592	5606	5621	5635	5650	5664	5678	5693	5707	5721	2	5	7	10	12
35	·5736	5750	5764	5779	5793	5807	5821	5835	5850	5864	2	5	7	9	12
36	·5878	5892	5906	5920	5934	5948	5962	5976	5990	6004	2	5	7	9	12
37	·6018	6032	6046	6060	6074	6088	6101	6115	6129	6143	2	5	7	9	12
38	·6157	6170	6184	6198	6211	6225	6239	6252	6266	6280	2	5	7	9	11
39	·6293	6307	6320	6334	6347	6361	6374	6388	6401	6414	2	4	7	9	11
40	·6428	6441	6455	6468	6481	6494	6508	6521	6534	6547	2	4	7	9	11
41	·6561	6574	6587	6600	6613	6626	6639	6652	6665	6678	2	4	7	9	11
42	·6691	6704	6717	6730	6743	6756	6769	6782	6794	6807	2	4	6	9	11
43	·6820	6833	6845	6858	6871	6884	6896	6909	6921	6934	2	4	6	8	11
44	·6947	6959	6972	6984	6997	7009	7022	7034	7046	7059	2	4	6	8	10

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
45°	·7071	7083	7096	7108	7120	7133	7145	7157	7169	7181	2	4	6	8	10
46	·7193	7206	7218	7230	7242	7254	7266	7278	7290	7302	2	4	6	8	10
47	·7314	7325	7337	7349	7361	7373	7385	7396	7408	7420	2	4	6	8	10
48	·7431	7443	7455	7466	7478	7490	7501	7513	7524	7536	2	4	6	8	10
49	·7547	7559	7570	7581	7593	7604	7615	7627	7638	7649	2	4	6	8	9
50	·7660	7672	7683	7694	7705	7716	7727	7738	7749	7760	2	4	6	7	9
51	·7771	7782	7793	7804	7815	7826	7837	7848	7859	7869	2	4	5	7	9
52	·7880	7891	7902	7912	7923	7934	7944	7955	7965	7976	2	4	5	7	9
53	·7986	7997	8007	8018	8028	8039	8049	8059	8070	8080	2	3	5	7	9
54	·8090	8100	8111	8121	8131	8141	8151	8161	8171	8181	2	3	5	7	8
55	·8192	8202	8211	8221	8231	8241	8251	8261	8271	8281	2	3	5	7	8
56	·8290	8300	8310	8320	8329	8339	8348	8358	8368	8377	2	3	5	6	8
57	·8387	8396	8406	8415	8425	8434	8443	8453	8462	8471	2	3	5	6	8
58	·8480	8490	8499	8508	8517	8526	8536	8545	8554	8563	2	3	5	6	8
59	·8572	8581	8590	8599	8607	8616	8625	8634	8643	8652	1	3	4	6	7
60	·8660	8669	8678	8686	8695	8704	8712	8721	8729	8738	1	3	4	6	7
61	·8746	8755	8763	8771	8780	8788	8796	8805	8813	8821	1	3	4	6	7
62	·8829	8838	8846	8854	8862	8870	8878	8886	8894	8902	1	3	4	5	7
63	·8910	8918	8926	8934	8942	8949	8957	8965	8973	8980	1	3	4	5	6
64	·8988	8996	9003	9011	9018	9026	9033	9041	9048	9056	1	3	4	5	6
65	·9063	9070	9078	9085	9092	9100	9107	9114	9121	9128	1	2	4	5	6
66	·9135	9143	9150	9157	9164	9171	9178	9184	9191	9198	1	2	3	5	6
67	·9205	9212	9219	9225	9232	9239	9245	9252	9259	9265	1	2	3	4	6
68	·9272	9278	9285	9291	9298	9304	9311	9317	9323	9330	1	2	3	4	5
69	·9336	9342	9348	9354	9361	9367	9373	9379	9385	9391	1	2	3	4	5
70	·9397	9403	9409	9415	9421	9426	9432	9438	9444	9449	1	2	3	4	5
71	·9455	9461	9466	9472	9478	9483	9489	9494	9500	9505	1	2	3	4	5
72	·9511	9516	9521	9527	9532	9537	9542	9548	9553	9558	1	2	3	3	4
73	·9563	9568	9573	9578	9583	9588	9593	9598	9603	9608	1	2	2	3	4
74	·9613	9617	9622	9627	9632	9636	9641	9646	9650	9655	1	2	2	3	4
75	·9659	9664	9668	9673	9677	9681	9686	9690	9694	9699	1	1	2	3	4
76	·9703	9707	9711	9715	9720	9724	9728	9732	9736	9740	1	1	2	3	3
77	·9744	9748	9751	9755	9759	9763	9767	9770	9774	9778	1	1	2	3	3
78	·9781	9785	9789	9792	9796	9799	9803	9806	9810	9813	1	1	2	2	3
79	·9816	9820	9823	9826	9829	9833	9836	9839	9842	9845	1	1	2	2	3
80	·9848	9851	9854	9857	9860	9863	9866	9869	9871	9874	0	1	1	2	2
81	·9877	9880	9882	9885	9888	9890	9893	9895	9898	9900	0	1	1	2	2
82	·9903	9905	9907	9910	9912	9914	9917	9919	9921	9923	0	1	1	2	2
83	·9925	9928	9930	9932	9934	9936	9938	9940	9942	9943	0	1	1	1	2
84	·9945	9947	9949	9951	9952	9954	9956	9957	9959	9960					
85	·9962	9963	9965	9966	9968	9969	9971	9972	9973	9974					
86	·9976	9977	9978	9979	9980	9981	9982	9983	9984	9985					
87	·9986	9987	9988	9989	9990	9990	9991	9992	9993	9993	Use Interpolation.				
88	·9994	9995	9995	9996	9996	9997	9997	9997	9998	9998					
89	·9998	9999	9999	9999	9999	1·000	1·000	1·000	1·000	1·000					

SUBTRACT

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0°	1·0000	1·000	1·000	1·000	1·000	1·000	*9999*	*9999*	*9999*	*9999*	Use Interpolation.				
1	·9998	9998	9998	9997	9997	9997	9996	9996	9995	9995					
2	·9994	9993	9993	9992	9991	9990	9990	9989	9988	9987					
3	·9986	9985	9984	9983	9982	9981	9980	9979	9978	9977					
4	·9976	9974	9973	9972	9971	9969	9968	9966	9965	9963					
5	·9962	9960	9959	9957	9956	9954	9952	9951	9949	9947					
6	·9945	9943	9942	9940	9938	9936	9934	9932	9930	9928	0	1	1	1	2
7	·9925	9923	9921	9919	9917	9914	9912	9910	9907	9905	0	1	1	2	2
8	·9903	9900	9898	9895	9893	9890	9888	9885	9882	9880	0	1	1	2	2
9	·9877	9874	9871	9869	9866	9863	9860	9857	9854	9851	0	1	1	2	2
10	·9848	9845	9842	9839	9836	9833	9829	9826	9823	9820	1	1	2	2	3
11	·9816	9813	9810	9806	9803	9799	9796	9792	9789	9785	1	1	2	2	3
12	·9781	9778	9774	9770	9767	9763	9759	9755	9751	9748	1	1	2	3	3
13	·9744	9740	9736	9732	9728	9724	9720	9715	9711	9707	1	1	2	3	3
14	·9703	9699	9694	9690	9686	9681	9677	9673	9668	9664	1	1	2	3	4
15	·9659	9655	9650	9646	9641	9636	9632	9627	9622	9617	1	2	2	3	4
16	·9613	9608	9603	9598	9593	9588	9583	9578	9573	9568	1	2	2	3	4
17	·9563	9558	9553	9548	9542	9537	9532	9527	9521	9516	1	2	3	3	4
18	·9511	9505	9500	9494	9489	9483	9478	9472	9466	9461	1	2	3	4	5
19	·9455	9449	9444	9438	9432	9426	9421	9415	9409	9403	1	2	3	4	5
20	·9397	9391	9385	9379	9373	9367	9361	9354	9348	9342	1	2	3	4	5
21	·9336	9330	9323	9317	9311	9304	9298	9291	9285	9278	1	2	3	4	5
22	·9272	9265	9259	9252	9245	9239	9232	9225	9219	9212	1	2	3	4	6
23	·9205	9198	9191	9184	9178	9171	9164	9157	9150	9143	1	2	3	5	6
24	·9135	9128	9121	9114	9107	9100	9092	9085	9078	9070	1	2	4	5	6
25	·9063	9056	9048	9041	9033	9026	9018	9011	9003	8996	1	3	4	5	6
26	·8988	8980	8973	8965	8957	8949	8942	8934	8926	8918	1	3	4	5	6
27	·8910	8902	8894	8886	8878	8870	8862	8854	8846	8838	1	3	4	5	7
28	·8829	8821	8813	8805	8796	8788	8780	8771	8763	8755	1	3	4	6	7
29	·8746	8738	8729	8721	8712	8704	8695	8686	8678	8669	1	3	4	6	7
30	·8660	8652	8643	8634	8625	8616	8607	8599	8590	8581	1	3	4	6	7
31	·8572	8563	8554	8545	8536	8526	8517	8508	8499	8490	2	3	5	6	8
32	·8480	8471	8462	8453	8443	8434	8425	8415	8406	8396	2	3	5	6	8
33	·8387	8377	8368	8358	8348	8339	8329	8320	8310	8300	2	3	5	6	8
34	·8290	8281	8271	8261	8251	8241	8231	8221	8211	8202	2	3	5	7	8
35	·8192	8181	8171	8161	8151	8141	8131	8121	8111	8100	2	3	5	7	8
36	·8090	8080	8070	8059	8049	8039	8028	8018	8007	7997	2	3	5	7	9
37	·7986	7976	7965	7955	7944	7934	7923	7912	7902	7891	2	4	5	7	9
38	·7880	7869	7859	7848	7837	7826	7815	7804	7793	7782	2	4	5	7	9
39	·7771	7760	7749	7738	7727	7716	7705	7694	7683	7672	2	4	6	7	9
40	·7660	7649	7638	7627	7615	7604	7593	7581	7570	7559	2	4	6	8	9
41	·7547	7536	7524	7513	7501	7490	7478	7466	7455	7443	2	4	6	8	10
42	·7431	7420	7408	7396	7385	7373	7361	7349	7337	7325	2	4	6	8	10
43	·7314	7302	7290	7278	7266	7254	7242	7230	7218	7206	2	4	6	8	10
44	·7193	7181	7169	7157	7145	7133	7120	7108	7096	7083	2	4	6	8	10

SUBTRACT

Where the integer changes, the numbers are italicised.

SUBTRACT

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
45°	·7071	7059	7046	7034	7022	7009	6997	6984	6972	6959	2	4	6	8	10
46	·6947	6934	6921	6909	6896	6884	6871	6858	6845	6833	2	4	6	8	11
47	·6820	6807	6794	6782	6769	6756	6743	6730	6717	6704	2	4	6	9	11
48	·6691	6678	6665	6652	6639	6626	6613	6600	6587	6574	2	4	7	9	11
49	·6561	6547	6534	6521	6508	6494	6481	6468	6455	6441	2	4	7	9	11
50	·6428	6414	6401	6388	6374	6361	6347	6334	6320	6307	2	4	7	9	11
51	·6293	6280	6266	6252	6239	6225	6211	6198	6184	6170	2	5	7	9	11
52	·6157	6143	6129	6115	6101	6088	6074	6060	6046	6032	2	5	7	9	12
53	·6018	6004	5990	5976	5962	5948	5934	5920	5906	5892	2	5	7	9	12
54	·5878	5864	5850	5835	5821	5807	5793	5779	5764	5750	2	5	7	9	12
55	·5736	5721	5707	5693	5678	5664	5650	5635	5621	5606	2	5	7	10	12
56	·5592	5577	5563	5548	5534	5519	5505	5490	5476	5461	2	5	7	10	12
57	·5446	5432	5417	5402	5388	5373	5358	5344	5329	5314	2	5	7	10	12
58	·5299	5284	5270	5255	5240	5225	5210	5195	5180	5165	2	5	7	10	12
59	·5150	5135	5120	5105	5090	5075	5060	5045	5030	5015	3	5	8	10	13
60	·5000	4985	4970	4955	4939	4924	4909	4894	4879	4863	3	5	8	10	13
61	·4848	4833	4818	4802	4787	4772	4756	4741	4726	4710	3	5	8	10	13
62	·4695	4679	4664	4648	4633	4617	4602	4586	4571	4555	3	5	8	10	13
63	·4540	4524	4509	4493	4478	4462	4446	4431	4415	4399	3	5	8	10	13
64	·4384	4368	4352	4337	4321	4305	4289	4274	4258	4242	3	5	8	11	13
65	·4226	4210	4195	4179	4163	4147	4131	4115	4099	4083	3	5	8	11	13
66	·4067	4051	4035	4019	4003	3987	3971	3955	3939	3923	3	5	8	11	13
67	·3907	3891	3875	3859	3843	3827	3811	3795	3778	3762	3	5	8	11	13
68	·3746	3730	3714	3697	3681	3665	3649	3633	3616	3600	3	5	8	11	14
69	·3584	3567	3551	3535	3518	3502	3486	3469	3453	3437	3	5	8	11	14
70	·3420	3404	3387	3371	3355	3338	3322	3305	3289	3272	3	5	8	11	14
71	·3256	3239	3223	3206	3190	3173	3156	3140	3123	3107	3	6	8	11	14
72	·3090	3074	3057	3040	3024	3007	2990	2974	2957	2940	3	6	8	11	14
73	·2924	2907	2890	2874	2857	2840	2823	2807	2790	2773	3	6	8	11	14
74	·2756	2740	2723	2706	2689	2672	2656	2639	2622	2605	3	6	8	11	14
75	·2588	2571	2554	2538	2521	2504	2487	2470	2453	2436	3	6	8	11	14
76	·2419	2402	2385	2368	2351	2334	2317	2300	2284	2267	3	6	8	11	14
77	·2250	2233	2215	2198	2181	2164	2147	2130	2113	2096	3	6	9	11	14
78	·2079	2062	2045	2028	2011	1994	1977	1959	1942	1925	3	6	9	11	14
79	·1908	1891	1874	1857	1840	1822	1805	1788	1771	1754	3	6	9	11	14
80	·1736	1719	1702	1685	1668	1650	1633	1616	1599	1582	3	6	9	11	14
81	·1564	1547	1530	1513	1495	1478	1461	1444	1426	1409	3	6	9	12	14
82	·1392	1374	1357	1340	1323	1305	1288	1271	1253	1236	3	6	9	12	14
83	·1219	1201	1184	1167	1149	1132	1115	1097	1080	1063	3	6	9	12	14
84	·1045	1028	1011	0993	0976	0958	0941	0924	0906	0889	3	6	9	12	14
85	·0872	0854	0837	0819	0802	0785	0767	0750	0732	0715	3	6	9	12	14
86	·0698	0680	0663	0645	0628	0610	0593	0576	0558	0541	3	6	9	12	15
87	·0523	0506	0488	0471	0454	0436	0419	0401	0384	0366	3	6	9	12	15
88	·0349	0332	0314	0297	0279	0262	0244	0227	0209	0192	3	6	9	12	15
89	·0175	0157	0140	0122	0105	0087	0070	0052	0035	0017	3	6	9	12	15

SUBTRACT

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0°	0·0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	15
1	0·0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	0·0349	0367	0384	0402	0419	0437	0454	0472	0489	0507	3	6	9	12	15
3	0·0524	0542	0559	0577	0594	0612	0629	0647	0664	0682	3	6	9	12	15
4	0·0699	0717	0734	0752	0769	0787	0805	0822	0840	0857	3	6	9	12	15
5	0·0875	0892	0910	0928	0945	0963	0981	0998	1016	1033	3	6	9	12	15
6	0·1051	1069	1086	1104	1122	1139	1157	1175	1192	1210	3	6	9	12	15
7	0·1228	1246	1263	1281	1299	1317	1334	1352	1370	1388	3	6	9	12	15
8	0·1405	1423	1441	1459	1477	1495	1512	1530	1548	1566	3	6	9	12	15
9	0·1584	1602	1620	1638	1655	1673	1691	1709	1727	1745	3	6	9	12	15
10	0·1763	1781	1799	1817	1835	1853	1871	1890	1908	1926	3	6	9	12	15
11	0·1944	1962	1980	1998	2016	2035	2053	2071	2089	2107	3	6	9	12	15
12	0·2126	2144	2162	2180	2199	2217	2235	2254	2272	2290	3	6	9	12	15
13	0·2309	2327	2345	2364	2382	2401	2419	2438	2456	2475	3	6	9	12	15
14	0·2493	2512	2530	2549	2568	2586	2605	2623	2642	2661	3	6	9	12	16
15	0·2679	2698	2717	2736	2754	2773	2792	2811	2830	2849	3	6	9	13	16
16	0·2867	2886	2905	2924	2943	2962	2981	3000	3019	3038	3	6	9	13	16
17	0·3057	3076	3096	3115	3134	3153	3172	3191	3211	3230	3	6	10	13	16
18	0·3249	3269	3288	3307	3327	3346	3365	3385	3404	3424	3	6	10	13	16
19	0·3443	3463	3482	3502	3522	3541	3561	3581	3600	3620	3	7	10	13	16
20	0·3640	3659	3679	3699	3719	3739	3759	3779	3799	3819	3	7	10	13	17
21	0·3839	3859	3879	3899	3919	3939	3959	3979	4000	4020	3	7	10	13	17
22	0·4040	4061	4081	4101	4122	4142	4163	4183	4204	4224	3	7	10	14	17
23	0·4245	4265	4286	4307	4327	4348	4369	4390	4411	4431	3	7	10	14	17
24	0·4452	4473	4494	4515	4536	4557	4578	4599	4621	4642	4	7	11	14	18
25	0·4663	4684	4706	4727	4748	4770	4791	4813	4834	4856	4	7	11	14	18
26	0·4877	4899	4921	4942	4964	4986	5008	5029	5051	5073	4	7	11	15	18
27	0·5095	5117	5139	5161	5184	5206	5228	5250	5272	5295	4	7	11	15	18
28	0·5317	5340	5362	5384	5407	5430	5452	5475	5498	5520	4	8	11	15	19
29	0·5543	5566	5589	5612	5635	5658	5681	5704	5727	5750	4	8	12	15	19
30	0·5774	5797	5820	5844	5867	5890	5914	5938	5961	5985	4	8	12	16	20
31	0·6009	6032	6056	6080	6104	6128	6152	6176	6200	6224	4	8	12	16	20
32	0·6249	6273	6297	6322	6346	6371	6395	6420	6445	6469	4	8	12	16	20
33	0·6494	6519	6544	6569	6594	6619	6644	6669	6694	6720	4	8	13	17	21
34	0·6745	6771	6796	6822	6847	6873	6899	6924	6950	6976	4	9	13	17	21
35	0·7002	7028	7054	7080	7107	7133	7159	7186	7212	7239	4	9	13	18	22
36	0·7265	7292	7319	7346	7373	7400	7427	7454	7481	7508	5	9	14	18	23
37	0·7536	7563	7590	7618	7646	7673	7701	7729	7757	7785	5	9	14	18	23
38	0·7813	7841	7869	7898	7926	7954	7983	8012	8040	8069	5	9	14	19	24
39	0·8098	8127	8156	8185	8214	8243	8273	8302	8332	8361	5	10	15	20	24
40	0·8391	8421	8451	8481	8511	8541	8571	8601	8632	8662	5	10	15	20	25
41	0·8693	8724	8754	8785	8816	8847	8878	8910	8941	8972	5	10	16	21	26
42	0·9004	9036	9067	9099	9131	9163	9195	9228	9260	9293	5	11	16	21	27
43	0·9325	9358	9391	9424	9457	9490	9523	9556	9590	9623	6	11	17	22	28
44	0·9657	9691	9725	9759	9793	9827	9861	9896	9930	9965	6	11	17	23	29
45	1·0000	0035	0070	0105	0141	0176	0212	0247	0283	0319	6	12	18	24	30
46	1·0355	0392	0428	0464	0501	0538	0575	0612	0649	0686	6	12	18	25	31
47	1·0724	0761	0799	0837	0875	0913	0951	0990	1028	1067	6	13	19	25	32

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
48°	1·1106	1145	1184	1224	1263	1303	1343	1383	1423	1463	7	13	20	27	33
49	1·1504	1544	1585	1626	1667	1708	1750	1792	1833	1875	7	14	21	28	34
50	1·1918	1960	2002	2045	2088	2131	2174	2218	2261	2305	7	14	22	29	36
51	1·2349	2393	2437	2482	2527	2572	2617	2662	2708	2753	8	15	23	30	38
52	1·2799	2846	2892	2938	2985	3032	3079	3127	3175	3222	8	16	24	31	39
53	1·3270	3319	3367	3416	3465	3514	3564	3613	3663	3713	8	16	25	33	41
54	1·3764	3814	3865	3916	3968	4019	4071	4124	4176	4229	9	17	26	35	43
55	1·4281	4335	4388	4442	4496	4550	4605	4659	4715	4770	9	18	27	36	45
56	1·4826	4882	4938	4994	5051	5108	5166	5224	5282	5340	10	19	29	38	48
57	1·5399	5458	5517	5577	5637	5697	5757	5818	5880	5941	10	20	30	40	50
58	1·6003	6066	6128	6191	6255	6319	6383	6447	6512	6577	11	21	32	43	53
59	1·6643	6709	6775	6842	6909	6977	7045	7113	7182	7251	11	23	34	45	56
60	1·7321	7391	7461	7532	7603	7675	7747	7820	7893	7966	12	24	36	48	60
61	1·8040	8115	8190	8265	8341	8418	8495	8572	8650	8728	13	26	38	51	64
62	1·8807	8887	8967	9047	9128	9210	9292	9375	9458	9542	14	27	41	55	68
63	1·9626	9711	9797	9883	9970	*0057*	*0145*	*0233*	*0323*	*0413*	15	29	44	58	73
64	2·0503	0594	0686	0778	0872	0965	1060	1155	1251	1348	16	31	47	63	78
65	2·1445	1543	1642	1742	1842	1943	2045	2148	2251	2355	17	34	51	68	85
66	2·2460	2566	2673	2781	2889	2998	3109	3220	3332	3445	18	37	55	73	92
67	2·3559	3673	3789	3906	4023	4142	4262	4383	4504	4627	20	40	60	79	99
											Difference for 1′				
											1 *to* 11	13 *to* 23	25 *to* 35	37 *to* 47	49 *to* 59
68	2·4751	4876	5002	5129	5257	5386	5517	5649	5782	5916	21	21	22	22	22
69	2·6051	6187	6325	6464	6605	6746	6889	7034	7179	7326	23	23	24	24	25
70	2·7475	7625	7776	7929	8083	8239	8397	8556	8716	8878	25	26	26	27	27
71	2·9042	9208	9375	9544	9714	9887	*0061*	*0237*	*0415*	*0595*	28	28	29	30	30
72	3·0777	0961	1146	1334	1524	1716	1910	2106	2305	2506	31	31	32	33	34
73	3·2709	2914	3122	3332	3544	3759	3977	4197	4420	4646	34	35	36	37	38
74	3·4874	5105	5339	5576	5816	6059	6305	6554	6806	7062	39	40	41	42	43
75	3·7321	7583	7848	8118	8391	8667	8947	9232	9520	9812	44	45	46	48	49
76	4·0108	0408	0713	1022	1335	1653	1976	2303	2635	2972	50	52	53	55	57
77	4·3315	3662	4015	4373	4737	5107	5483	5864	6252	6646	58	60	62	64	66
78	4·7046	7453	7867	8288	8716	9152	9594	*0045*	*0504*	*0970*	68	71	73	76	78
79	5·1446	1929	2422	2924	3435	3955	4486	5026	5578	6140	81	84	88	91	95
80	5·671	5·730	5·789	5·850	5·912	5·976	6·041	6·107	6·174	6·243	10	10	11	11	12
81	6·314	6·386	6·460	6·535	6·612	6·691	6·772	6·855	6·940	7·026	12	13	13	14	15
82	7·115	7·207	7·300	7·396	7·495	7·596	7·700	7·806	7·916	8·028	15	16	17	18	19
83	8·144	8·264	8·386	8·513	8·643	8·777	8·915	9·058	9·205	9·357	20	21	23	24	26
84	9·51	9·68	9·84	10·02	10·20	10·39	10·58	10·78	10·99	11·20	3	3	3	3	4
85	11·43	11·66	11·91	12·16	12·43	12·71	13·00	13·30	13·62	13·95	4	4	5	5	6
86	14·30	14·67	15·06	15·46	15·89	16·35	16·83	17·34	17·89	18·46	6	7	8	9	10
87	19·08	19·74	20·45	21·20	22·02	22·90	23·86	24·90	26·03	27·27	11	13	15	18	22
88	28·64	30·14	31·82	33·69	35·80	38·19	40·92	44·07	47·74	52·08					
89	57·29	63·66	71·62	81·85	95·49	114·6	143·2	191·0	286·5	573·0					

Where the integer changes, the numbers are italicised.

NATURAL COTANGENTS

SUBTRACT

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Difference for 1′: 1 to 11	13 to 23	25 to 35	37 to 47	49 to 59
0°	∞	573·0	286·5	191·0	143·2	114·6	95·49	81·85	71·62	63·66					
1	57·29	52·08	47·74	44·07	40·92	38·19	35·80	33·69	31·82	30·14	22	18	15	13	11
2	28·64	27·27	26·03	24·90	23·86	22·90	22·02	21·20	20·45	19·74	10	9	8	7	6
3	19·08	18·46	17·89	17·34	16·83	16·35	15·89	15·46	15·06	14·67					
4	14·30	13·95	13·62	13·30	13·00	12·71	12·43	12·16	11·91	11·66	6	5	5	4	4
5	11·43	11·20	10·99	10·78	10·58	10·39	10·20	10·02	9·84	9·68	4	3	3	3	3
6	9·514	9·357	9·205	9·058	8·915	8·777	8·643	8·513	8·386	8·264	26	24	23	21	20
7	8·144	8·028	7·916	7·806	7·700	7·596	7·495	7·396	7·300	7·207	19	18	17	16	15
8	7·115	7·026	6·940	6·855	6·772	6·691	6·612	6·535	6·460	6·386	15	14	13	13	12
9	6·314	6·243	6·174	6·107	6·041	5·976	5·912	5·850	5·789	5·730	12	11	11	10	10
10	5·6713	6140	5578	5026	4486	3955	3435	2924	2422	1929	95	91	88	84	81
11	5·1446	0970	0504	0045	*9594*	*9152*	*8716*	*8288*	*7867*	*7453*	78	76	73	71	68
12	4·7046	6646	6252	5864	5483	5107	4737	4373	4015	3662	66	64	62	60	58
13	4·3315	2972	2635	2303	1976	1653	1335	1022	0713	0408	57	55	53	52	50
14	4·0108	*9812*	*9520*	*9232*	*8947*	*8667*	*8391*	*8118*	*7848*	*7583*	49	48	46	45	44
15	3·7321	7062	6806	6554	6305	6059	5816	5576	5339	5105	43	42	41	40	39
16	3·4874	4646	4420	4197	3977	3759	3544	3332	3122	2914	38	37	36	35	34
17	3·2709	2506	2305	2106	1910	1716	1524	1334	1146	0961	34	33	32	31	31
18	3·0777	0595	0415	0237	0061	*9887*	*9714*	*9544*	*9375*	*9208*	30	30	29	28	28
19	2·9042	8878	8716	8556	8397	8239	8083	7929	7776	7625	27	27	26	26	25
20	2·7475	7326	7179	7034	6889	6746	6605	6464	6325	6187	25	24	24	23	23
21	2·6051	5916	5782	5649	5517	5386	5257	5129	5002	4876	22	22	22	21	21
											1′	2′	3′	4′	5′
22	2·4751	4627	4504	4383	4262	4142	4023	3906	3789	3673	20	40	60	79	99
23	2·3559	3445	3332	3220	3109	2998	2889	2781	2673	2566	18	37	55	73	92
24	2·2460	2355	2251	2148	2045	1943	1842	1742	1642	1543	17	34	51	68	85
25	2·1445	1348	1251	1155	1060	0965	0872	0778	0686	0594	16	31	47	63	78
26	2·0503	0413	0323	0233	0145	0057	*9970*	*9883*	*9797*	*9711*	15	29	44	58	73
27	1·9626	9542	9458	9375	9292	9210	9128	9047	8967	8887	14	27	41	55	68
28	1·8807	8728	8650	8572	8495	8418	8341	8265	8190	8115	13	26	38	51	64
29	1·8040	7966	7893	7820	7747	7675	7603	7532	7461	7391	12	24	36	48	60
30	1·7321	7251	7182	7113	7045	6977	6909	6842	6775	6709	11	23	34	45	56
31	1·6643	6577	6512	6447	6383	6319	6255	6191	6128	6066	11	21	32	43	53
32	1·6003	5941	5880	5818	5757	5697	5637	5577	5517	5458	10	20	30	40	50
33	1·5399	5340	5282	5224	5166	5108	5051	4994	4938	4882	10	19	29	38	48
34	1·4826	4770	4715	4659	4605	4550	4496	4442	4388	4335	9	18	27	36	45
35	1·4281	4229	4176	4124	4071	4019	3968	3916	3865	3814	9	17	26	35	43
36	1·3764	3713	3663	3613	3564	3514	3465	3416	3367	3319	8	16	25	33	41
37	1·3270	3222	3175	3127	3079	3032	2985	2938	2892	2846	8	16	24	31	39
38	1·2799	2753	2708	2662	2617	2572	2527	2482	2437	2393	8	15	23	30	38
39	1·2349	2305	2261	2218	2174	2131	2088	2045	2002	1960	7	14	22	29	36
40	1·1918	1875	1833	1792	1750	1708	1667	1626	1585	1544	7	14	21	28	34
41	1·1504	1463	1423	1383	1343	1303	1263	1224	1184	1145	7	13	20	27	33
42	1·1106	1067	1028	0990	0951	0913	0875	0837	0799	0761	6	13	19	25	32
43	1·0724	0686	0649	0612	0575	0538	0501	0464	0428	0392	6	12	18	25	31

SUBTRACT

Where the integer changes, the numbers are italicised.

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
											SUBTRACT				
44°	1·0355	0319	0283	0247	0212	0176	0141	0105	0070	0035	6	12	18	24	30
45	1·0000	*9965*	*9930*	*9896*	*9861*	*9827*	*9793*	*9759*	*9725*	*9691*	6	11	17	23	29
46	0·9657	9623	9590	9556	9523	9490	9457	9424	9391	9358	6	11	17	22	28
47	0·9325	9293	9260	9228	9195	9163	9131	9099	9067	9036	5	11	16	21	27
48	0·9004	8972	8941	8910	8878	8847	8816	8785	8754	8724	5	10	16	21	26
49	0·8693	8662	8632	8601	8571	8541	8511	8481	8451	8421	5	10	15	20	25
50	0·8391	8361	8332	8302	8273	8243	8214	8185	8156	8127	5	10	15	20	24
51	0·8098	8069	8040	8012	7983	7954	7926	7898	7869	7841	5	9	14	19	24
52	0·7813	7785	7757	7729	7701	7673	7646	7618	7590	7563	5	9	14	18	23
53	0·7536	7508	7481	7454	7427	7400	7373	7346	7319	7292	5	9	14	18	23
54	0·7265	7239	7212	7186	7159	7133	7107	7080	7054	7028	4	9	13	18	22
55	0·7002	6976	6950	6924	6899	6873	6847	6822	6796	6771	4	9	13	17	21
56	0·6745	6720	6694	6669	6644	6619	6594	6569	6544	6519	4	8	13	17	21
57	0·6494	6469	6445	6420	6395	6371	6346	6322	6297	6273	4	8	12	16	20
58	0·6249	6224	6200	6176	6152	6128	6104	6080	6056	6032	4	8	12	16	20
59	0·6009	5985	5961	5938	5914	5890	5867	5844	5820	5797	4	8	12	16	20
60	0·5774	5750	5727	5704	5681	5658	5635	5612	5589	5566	4	8	12	15	19
61	0·5543	5520	5498	5475	5452	5430	5407	5384	5362	5340	4	8	11	15	19
62	0·5317	5295	5272	5250	5228	5206	5184	5161	5139	5117	4	7	11	15	18
63	0·5095	5073	5051	5029	5008	4986	4964	4942	4921	4899	4	7	11	15	18
64	0·4877	4856	4834	4813	4791	4770	4748	4727	4706	4684	4	7	11	14	18
65	0·4663	4642	4621	4599	4578	4557	4536	4515	4494	4473	4	7	11	14	18
66	0·4452	4431	4411	4390	4369	4348	4327	4307	4286	4265	3	7	10	14	17
67	0·4245	4224	4204	4183	4163	4142	4122	4101	4081	4061	3	7	10	14	17
68	0·4040	4020	4000	3979	3959	3939	3919	3899	3879	3859	3	7	10	13	17
69	0·3839	3819	3799	3779	3759	3739	3719	3699	3679	3659	3	7	10	13	17
70	0·3640	3620	3600	3581	3561	3541	3522	3502	3482	3463	3	7	10	13	16
71	0·3443	3424	3404	3385	3365	3346	3327	3307	3288	3269	3	6	10	13	16
72	0·3249	3230	3211	3191	3172	3153	3134	3115	3096	3076	3	6	10	13	16
73	0·3057	3038	3019	3000	2981	2962	2943	2924	2905	2886	3	6	9	13	16
74	0·2867	2849	2830	2811	2792	2773	2754	2736	2717	2698	3	6	9	13	16
75	0·2679	2661	2642	2623	2605	2586	2568	2549	2530	2512	3	6	9	12	16
76	0·2493	2475	2456	2438	2419	2401	2382	2364	2345	2327	3	6	9	12	15
77	0·2309	2290	2272	2254	2235	2217	2199	2180	2162	2144	3	6	9	12	15
78	0·2126	2107	2089	2071	2053	2035	2016	1998	1980	1962	3	6	9	12	15
79	0·1944	1926	1908	1890	1871	1853	1835	1817	1799	1781	3	6	9	12	15
80	0·1763	1745	1727	1709	1691	1673	1655	1638	1620	1602	3	6	9	12	15
81	0·1584	1566	1548	1530	1512	1495	1477	1459	1441	1423	3	6	9	12	15
82	0·1405	1388	1370	1352	1334	1317	1299	1281	1263	1246	3	6	9	12	15
83	0·1228	1210	1192	1175	1157	1139	1122	1104	1086	1069	3	6	9	12	15
84	0·1051	1033	1016	0998	0981	0963	0945	0928	0910	0892	3	6	9	12	15
85	0·0875	0857	0840	0822	0805	0787	0769	0752	0734	0717	3	6	9	12	15
86	0·0699	0682	0664	0647	0629	0612	0594	0577	0559	0542	3	6	9	12	15
87	0·0524	0507	0489	0472	0454	0437	0419	0402	0384	0367	3	6	9	12	15
88	0·0349	0332	0314	0297	0279	0262	0244	0227	0209	0192	3	6	9	12	15
89	0·0175	0157	0140	0122	0105	0087	0070	0052	0035	0017	3	6	9	12	15
											SUBTRACT				

Where the integer changes, the numbers are italicised.

NATURAL COSECANTS

SUBTRACT

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	Difference for 1′ 1 to 11	13 to 23	25 to 35	37 to 47	49 to 59
0°	∞	573·0	286·5	191·0	143·2	114·6	95·49	81·85	71·62	63·66					
1	57·30	52·09	47·75	44·08	40·93	38·20	35·81	33·71	31·84	30·16	—	—	—	—	—
2	28·65	27·29	26·05	24·92	23·88	22·93	22·04	21·23	20·47	19·77	22	18	15	13	11
3	19·11	18·49	17·91	17·37	16·86	16·38	15·93	15·50	15·09	14·70	10	9	8	7	6
4	14·34	13·99	13·65	13·34	13·03	12·75	12·47	12·20	11·95	11·71	6	5	5	4	4
5	11·47	11·25	11·03	10·83	10·63	10·43	10·25	10·07	9·90	9·73	4	3	3	3	3
6	9·567	9·411	9·259	9·113	8·971	8·834	8·700	8·571	8·446	8·324	26	24	23	21	20
7	8·206	8·091	7·979	7·870	7·764	7·661	7·561	7·463	7·368	7·276	19	18	17	16	15
8	7·185	7·097	7·011	6·927	6·845	6·765	6·687	6·611	6·537	6·464	14	14	13	13	12
9	6·392	6·323	6·255	6·188	6·123	6·059	5·996	5·935	5·875	5·816	11	11	11	10	10
10	5·7588	7023	6470	5928	5396	4874	4362	3860	3367	2883	93	90	86	83	80
11	5·2408	1942	1484	1034	0593	0159	*9732*	*9313*	*8901*	*8496*	77	74	72	69	67
12	4·8097	7706	7321	6942	6569	6202	5841	5486	5137	4793	65	63	61	59	57
13	4·4454	4121	3792	3469	3150	2837	2527	2223	1923	1627	55	53	52	50	49
14	4·1336	1048	0765	0486	0211	*9939*	*9672*	*9408*	*9147*	*8890*	48	46	45	44	43
15	3·8637	8387	8140	7897	7657	7420	7186	6955	6727	6502	41	40	39	38	37
16	3·6280	6060	5843	5629	5418	5209	5003	4799	4598	4399	36	35	35	34	33
17	3·4203	4009	3817	3628	3440	3255	3072	2891	2712	2535	32	31	31	30	29
18	3·2361	2188	2017	1848	1681	1515	1352	1190	1030	0872	29	28	27	27	26
19	3·0716	0561	0407	0256	0106	*9957*	*9811*	*9665*	*9521*	*9379*	26	25	25	24	24
20	2·9238	9099	8960	8824	8688	8555	8422	8291	8161	8032	23	23	22	22	21
21	2·7904	7778	7653	7529	7407	7285	7165	7046	6927	6811	21	21	20	20	19
											1′	2′	3′	4′	5′
22	2·6695	6580	6466	6354	6242	6131	6022	5913	5805	5699	18	37	55	73	92
23	2·5593	5488	5384	5282	5180	5078	4978	4879	4780	4683	17	34	50	67	84
24	2·4586	4490	4395	4300	4207	4114	4022	3931	3841	3751	15	31	46	62	77
25	2·3662	3574	3486	3400	3314	3228	3144	3060	2976	2894	14	28	43	57	71
26	2·2812	2730	2650	2570	2490	2412	2333	2256	2179	2103	13	26	39	52	65
27	2·2027	1952	1877	1803	1730	1657	1584	1513	1441	1371	12	24	36	48	61
28	2·1301	1231	1162	1093	1025	0957	0890	0824	0757	0692	11	22	34	45	56
29	2·0627	0562	0498	0434	0371	0308	0245	0183	0122	0061	10	21	31	42	52
30	2·0000	*9940*	*9880*	*9821*	*9762*	*9703*	*9645*	*9587*	*9530*	*9473*	10	19	29	39	49
31	1·9416	9360	9304	9249	9194	9139	9084	9031	8977	8924	9	18	27	36	45
32	1·8871	8818	8766	8714	8663	8612	8561	8510	8460	8410	8	17	25	34	42
33	1·8361	8312	8263	8214	8166	8118	8070	8023	7976	7929	8	16	24	32	40
34	1·7883	7837	7791	7745	7700	7655	7610	7566	7522	7478	7	15	22	30	37
35	1·7434	7391	7348	7305	7263	7221	7179	7137	7095	7054	7	14	21	28	35
36	1·7013	6972	6932	6892	6852	6812	6772	6733	6694	6655	7	13	20	26	33
37	1·6616	6578	6540	6502	6464	6427	6390	6353	6316	6279	6	12	19	25	31
38	1·6243	6207	6171	6135	6099	6064	6029	5994	5959	5925	6	12	18	23	29
39	1·5890	5856	5822	5788	5755	5721	5688	5655	5622	5590	6	11	17	22	28
40	1·5557	5525	5493	5461	5429	5398	5366	5335	5304	5273	5	10	16	21	26
41	1·5243	5212	5182	5151	5121	5092	5062	5032	5003	4974	5	10	15	20	25

SUBTRACT

Where the integer changes, the numbers are italicised.

SUBTRACT

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
42°	1·4945	4916	4887	4859	4830	4802	4774	4746	4718	4690	5	9	14	19	23
43	1·4663	4635	4608	4581	4554	4527	4501	4474	4448	4422	4	9	13	18	22
44	1·4396	4370	4344	4318	4293	4267	4242	4217	4192	4167	4	8	13	17	21
45	1·4142	4118	4093	4069	4044	4020	3996	3972	3949	3925	4	8	12	16	20
46	1·3902	3878	3855	3832	3809	3786	3763	3741	3718	3696	4	8	11	15	19
47	1·3673	3651	3629	3607	3585	3563	3542	3520	3499	3478	4	7	11	14	18
48	1·3456	3435	3414	3393	3373	3352	3331	3311	3291	3270	3	7	10	14	17
49	1·3250	3230	3210	3190	3171	3151	3131	3112	3093	3073	3	7	10	13	16
50	1·3054	3035	3016	2997	2978	2960	2941	2923	2904	2886	3	6	9	12	16
51	1·2868	2849	2831	2813	2796	2778	2760	2742	2725	2708	3	6	9	12	15
52	1·2690	2673	2656	2639	2622	2605	2588	2571	2554	2538	3	6	8	11	14
53	1·2521	2505	2489	2472	2456	2440	2424	2408	2392	2376	3	5	8	11	13
54	1·2361	2345	2329	2314	2299	2283	2268	2253	2238	2223	3	5	8	10	13
55	1·2208	2193	2178	2163	2149	2134	2120	2105	2091	2076	2	5	7	10	12
56	1·2062	2048	2034	2020	2006	1992	1978	1964	1951	1937	2	5	7	9	12
57	1·1924	1910	1897	1883	1870	1857	1844	1831	1818	1805	2	4	7	9	11
58	1·1792	1779	1766	1753	1741	1728	1716	1703	1691	1679	2	4	6	8	10
59	1·1666	1654	1642	1630	1618	1606	1594	1582	1570	1559	2	4	6	8	10
60	1·1547	1535	1524	1512	1501	1490	1478	1467	1456	1445	2	4	6	8	9
61	1·1434	1423	1412	1401	1390	1379	1368	1357	1347	1336	2	4	5	7	9
62	1·1326	1315	1305	1294	1284	1274	1264	1253	1243	1233	2	3	5	7	9
63	1·1223	1213	1203	1194	1184	1174	1164	1155	1145	1136	2	3	5	6	8
64	1·1126	1117	1107	1098	1089	1079	1070	1061	1052	1043	2	3	5	6	8
65	1·1034	1025	1016	1007	0998	0989	0981	0972	0963	0955	1	3	4	6	7
66	1·0946	0938	0929	0921	0913	0904	0896	0888	0880	0872	1	3	4	6	7
67	1·0864	0856	0848	0840	0832	0824	0816	0808	0801	0793	1	3	4	5	7
68	1·0785	0778	0770	0763	0755	0748	0740	0733	0726	0719	1	2	4	5	6
69	1·0711	0704	0697	0690	0683	0676	0669	0662	0655	0649	1	2	3	5	6
70	1·0642	0635	0628	0622	0615	0608	0602	0595	0589	0583	1	2	3	4	5
71	1·0576	0570	0564	0557	0551	0545	0539	0533	0527	0521	1	2	3	4	5
72	1·0515	0509	0503	0497	0491	0485	0480	0474	0468	0463	1	2	3	4	5
73	1·0457	0451	0446	0440	0435	0429	0424	0419	0413	0408	1	2	3	4	4
74	1·0403	0398	0393	0388	0382	0377	0372	0367	0363	0358	1	2	3	3	4
75	1·0353	0348	0343	0338	0334	0329	0324	0320	0315	0311	1	2	2	3	4
76	1·0306	0302	0297	0293	0288	0284	0280	0276	0271	0267	1	1	2	3	4
77	1·0263	0259	0255	0251	0247	0243	0239	0235	0231	0227	1	1	2	3	3
78	1·0223	0220	0216	0212	0209	0205	0201	0198	0194	0191	1	1	2	2	3
79	1·0187	0184	0180	0177	0174	0170	0167	0164	0161	0157	1	1	2	2	3
80	1·0154	0151	0148	0145	0142	0139	0136	0133	0130	0127	0	1	1	2	2
81	1·0125	0122	0119	0116	0114	0111	0108	0106	0103	0101	0	1	1	2	2
82	1·0098	0096	0093	0091	0089	0086	0084	0082	0079	0077	0	1	1	2	2
83	1·0075	0073	0071	0069	0067	0065	0063	0061	0059	0057	0	1	1	1	2
84	1·0055	0053	0051	0050	0048	0046	0045	0043	0041	0040					
85	1·0038	0037	0035	0034	0032	0031	0030	0028	0027	0026					
86	1·0024	0023	0022	0021	0020	0019	0018	0017	0016	0015					
87	1·0014	0013	0012	0011	0010	0010	0009	0008	0007	0007	Use Interpolation.				
88	1·0006	0006	0005	0004	0004	0003	0003	0003	0002	0002					
89	1·0002	0001	0001	0001	0001	0000	0000	0000	0000	0000					

SUBTRACT

Where the integer changes, the numbers are italicised.

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0°	1·0000	0000	0000	0000	0000	0000	0001	0001	0001	0001	Use Interpolation.				
1	1·0002	0002	0002	0003	0003	0003	0004	0004	0005	0006					
2	1·0006	0007	0007	0008	0009	0010	0010	0011	0012	0013					
3	1·0014	0015	0016	0017	0018	0019	0020	0021	0022	0023					
4	1·0024	0026	0027	0028	0030	0031	0032	0034	0035	0037					
5	1·0038	0040	0041	0043	0045	0046	0048	0050	0051	0053					
6	1·0055	0057	0059	0061	0063	0065	0067	0069	0071	0073	0	1	1	1	2
7	1·0075	0077	0079	0082	0084	0086	0089	0091	0093	0096	0	1	1	2	2
8	1·0098	0101	0103	0106	0108	0111	0114	0116	0119	0122	0	1	1	2	2
9	1·0125	0127	0130	0133	0136	0139	0142	0145	0148	0151	0	1	1	2	2
10	1·0154	0157	0161	0164	0167	0170	0174	0177	0180	0184	1	1	2	2	3
11	1·0187	0191	0194	0198	0201	0205	0209	0212	0216	0220	1	1	2	2	3
12	1·0223	0227	0231	0235	0239	0243	0247	0251	0255	0259	1	1	2	3	3
13	1·0263	0267	0271	0276	0280	0284	0288	0293	0297	0302	1	1	2	3	4
14	1·0306	0311	0315	0320	0324	0329	0334	0338	0343	0348	1	2	2	3	4
15	1·0353	0358	0363	0367	0372	0377	0382	0388	0393	0398	1	2	3	3	4
16	1·0403	0408	0413	0419	0424	0429	0435	0440	0446	0451	1	2	3	4	4
17	1·0457	0463	0468	0474	0480	0485	0491	0497	0503	0509	1	2	3	4	5
18	1·0515	0521	0527	0533	0539	0545	0551	0557	0564	0570	1	2	3	4	5
19	1·0576	0583	0589	0595	0602	0608	0615	0622	0628	0635	1	2	3	4	5
20	1·0642	0649	0655	0662	0669	0676	0683	0690	0697	0704	1	2	3	5	6
21	1·0711	0719	0726	0733	0740	0748	0755	0763	0770	0778	1	2	4	5	6
22	1·0785	0793	0801	0808	0816	0824	0832	0840	0848	0856	1	3	4	5	7
23	1·0864	0872	0880	0888	0896	0904	0913	0921	0929	0938	1	3	4	6	7
24	1·0946	0955	0963	0972	0981	0989	0998	1007	1016	1025	1	3	4	6	7
25	1·1034	1043	1052	1061	1070	1079	1089	1098	1107	1117	2	3	5	6	8
26	1·1126	1136	1145	1155	1164	1174	1184	1194	1203	1213	2	3	5	6	8
27	1·1223	1233	1243	1253	1264	1274	1284	1294	1305	1315	2	3	5	7	9
28	1·1326	1336	1347	1357	1368	1379	1390	1401	1412	1423	2	4	5	7	9
29	1·1434	1445	1456	1467	1478	1490	1501	1512	1524	1535	2	4	6	8	9
30	1·1547	1559	1570	1582	1594	1606	1618	1630	1642	1654	2	4	6	8	10
31	1·1666	1679	1691	1703	1716	1728	1741	1753	1766	1779	2	4	6	8	10
32	1·1792	1805	1818	1831	1844	1857	1870	1883	1897	1910	2	4	7	9	11
33	1·1924	1937	1951	1964	1978	1992	2006	2020	2034	2048	2	5	7	9	12
34	1·2062	2076	2091	2105	2120	2134	2149	2163	2178	2193	2	5	7	10	12
35	1·2208	2223	2238	2253	2268	2283	2299	2314	2329	2345	3	5	8	10	13
36	1·2361	2376	2392	2408	2424	2440	2456	2472	2489	2505	3	5	8	11	13
37	1·2521	2538	2554	2571	2588	2605	2622	2639	2656	2673	3	6	8	11	14
38	1·2690	2708	2725	2742	2760	2778	2796	2813	2831	2849	3	6	9	12	15
39	1·2868	2886	2904	2923	2941	2960	2978	2997	3016	3035	3	6	9	12	16
40	1·3054	3073	3093	3112	3131	3151	3171	3190	3210	3230	3	7	10	13	16
41	1·3250	3270	3291	3311	3331	3352	3373	3393	3414	3435	3	7	10	14	17
42	1·3456	3478	3499	3520	3542	3563	3585	3607	3629	3651	4	7	11	14	18
43	1·3673	3696	3718	3741	3763	3786	3809	3832	3855	3878	4	8	11	15	19
44	1·3902	3925	3949	3972	3996	4020	4044	4069	4093	4118	4	8	12	16	20
45	1·4142	4167	4192	4217	4242	4267	4293	4318	4344	4370	4	8	13	17	21
46	1·4396	4422	4448	4474	4501	4527	4554	4581	4608	4635	4	9	13	18	22
47	1·4663	4690	4718	4746	4774	4802	4830	4859	4887	4916	5	9	14	19	23

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
48°	1·4945	4974	5003	5032	5062	5092	5121	5151	5182	5212	5	10	15	20	25
49	1·5243	5273	5304	5335	5366	5398	5429	5461	5493	5525	5	10	16	21	26
50	1·5557	5590	5622	5655	5688	5721	5755	5788	5822	5856	6	11	17	22	28
51	1·5890	5925	5959	5994	6029	6064	6099	6135	6171	6207	6	12	18	23	29
52	1·6243	6279	6316	6353	6390	6427	6464	6502	6540	6578	6	12	19	25	31
53	1·6616	6655	6694	6733	6772	6812	6852	6892	6932	6972	7	13	20	26	33
54	1·7013	7054	7095	7137	7179	7221	7263	7305	7348	7391	7	14	21	28	35
55	1·7434	7478	7522	7566	7610	7655	7700	7745	7791	7837	7	15	22	30	37
56	1·7883	7929	7976	8023	8070	8118	8166	8214	8263	8312	8	16	24	32	40
57	1·8361	8410	8460	8510	8561	8612	8663	8714	8766	8818	8	17	25	34	42
58	1·8871	8924	8977	9031	9084	9139	9194	9249	9304	9360	9	18	27	36	45
59	1·9416	9473	9530	9587	9645	9703	9762	9821	9880	9940	10	19	29	39	49
60	2·0000	0061	0122	0183	0245	0308	0371	0434	0498	0562	10	21	31	42	52
61	2·0627	0692	0757	0824	0890	0957	1025	1093	1162	1231	11	22	34	45	56
62	2·1301	1371	1441	1513	1584	1657	1730	1803	1877	1952	12	24	36	48	61
63	2·2027	2103	2179	2256	2333	2412	2490	2570	2650	2730	13	26	39	52	65
64	2·2812	2894	2976	3060	3144	3228	3314	3400	3486	3574	14	28	43	57	71
65	2·3662	3751	3841	3931	4022	4114	4207	4300	4395	4490	15	31	46	62	77
66	2·4586	4683	4780	4879	4978	5078	5180	5282	5384	5488	17	34	50	67	84
67	2·5593	5699	5805	5913	6022	6131	6242	6354	6466	6580	18	37	55	73	92
											Difference for 1′				
											1 to 11	13 to 23	25 to 35	37 to 47	49 to 59
68	2·6695	6811	6927	7046	7165	7285	7407	7529	7653	7778	19	20	20	21	21
69	2·7904	8032	8161	8291	8422	8555	8688	8824	8960	9099	21	22	22	23	23
70	2·9238	9379	9521	9665	9811	9957	*0106*	*0256*	*0407*	*0561*	24	24	25	25	26
71	3·0716	0872	1030	1190	1352	1515	1681	1848	2017	2188	26	27	27	28	29
72	3·2361	2535	2712	2891	3072	3255	3440	3628	3817	4009	29	30	31	31	32
73	3·4203	4399	4598	4799	5003	5209	5418	5629	5843	6060	33	34	35	35	36
74	3·6280	6502	6727	6955	7186	7420	7657	7897	8140	8387	37	38	39	40	41
75	3·8637	8890	9147	9408	9672	9939	*0211*	*0486*	*0765*	*1048*	43	44	45	46	48
76	4·1336	1627	1923	2223	2527	2837	3150	3469	3792	4121	49	50	52	53	55
77	4·4454	4793	5137	5486	5841	6202	6569	6942	7321	7706	57	59	61	63	65
78	4·8097	8496	8901	9313	9732	*0159*	*0593*	*1034*	*1484*	*1942*	67	69	72	74	77
79	5·2408	2883	3367	3860	4362	4874	5396	5928	6470	7023	80	83	86	90	93
80	5·759	5·816	5·875	5·935	5·996	6·059	6·123	6·188	6·255	6·323	10	10	11	11	11
81	6·392	6·464	6·537	6·611	6·687	6·765	6·845	6·927	7·011	7·097	12	13	13	14	14
82	7·185	7·276	7·368	7·463	7·561	7·661	7·764	7·870	7·979	8·091	15	16	17	18	19
83	8·206	8·324	8·446	8·571	8·700	8·834	8·971	9·113	9·259	9·411	20	21	23	24	26
84	9·57	9·73	9·90	10·07	10·25	10·43	10·63	10·83	11·03	11·25	3	3	3	3	4
85	11·47	11·71	11·95	12·20	12·47	12·75	13·03	13·34	13·65	13·99	4	4	5	5	6
86	14·34	14·70	15·09	15·50	15·93	16·38	16·86	17·37	17·91	18·49	6	7	8	9	10
87	19·11	19·77	20·47	21·23	22·04	22·93	23·88	24·92	26·05	27·29	11	13	15	18	22
88	28·65	30·16	31·84	33·71	35·81	38·20	40·93	44·08	47·75	52·09					
89	57·30	63·66	71·62	81·85	95·49	114·6	143·2	191·0	286·5	573·0					

Where the integer changes, the numbers are italicised.

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′
0°	·00000	00175	00349	00524	00698	00873	01047	01222	01396	01571
1	·01745	01920	02094	02269	02443	02618	02793	02967	03142	03316
2	·03491	03665	03840	04014	04189	04363	04538	04712	04887	05061
3	·05236	05411	05585	05760	05934	06109	06283	06458	06632	06807
4	·06981	07156	07330	07505	07679	07854	08029	08203	08378	08552
5	·08727	08901	09076	09250	09425	09599	09774	09948	10123	10297
6	·10472	10647	10821	10996	11170	11345	11519	11694	11868	12043
7	·12217	12392	12566	12741	12915	13090	13265	13439	13614	13788
8	·13963	14137	14312	14486	14661	14835	15010	15184	15359	15533
9	·15708	15882	16057	16232	16406	16581	16755	16930	17104	17279
10	·17453	17628	17802	17977	18151	18326	18500	18675	18850	19024
11	·19199	19373	19548	19722	19897	20071	20246	20420	20595	20769
12	·20944	21118	21293	21468	21642	21817	21991	22166	22340	22515
13	·22689	22864	23038	23213	23387	23562	23736	23911	24086	24260
14	·24435	24609	24784	24958	25133	25307	25482	25656	25831	26005
15	·26180	26354	26529	26704	26878	27053	27227	27402	27576	27751
16	·27925	28100	28274	28449	28623	28798	28972	29147	29322	29496
17	·29671	29845	30020	30194	30369	30543	30718	30892	31067	31241
18	·31416	31590	31765	31940	32114	32289	32463	32638	32812	32987
19	·33161	33336	33510	33685	33859	34034	34208	34383	34558	34732
20	·34907	35081	35256	35430	35605	35779	35954	36128	36303	36477
21	·36652	36826	37001	37176	37350	37525	37699	37874	38048	38223
22	·38397	38572	38746	38921	39095	39270	39444	39619	39794	39968
23	·40143	40317	40492	40666	40841	41015	41190	41364	41539	41713
24	·41888	42062	42237	42411	42586	42761	42935	43110	43284	43459
25	·43633	43808	43982	44157	44331	44506	44680	44855	45029	45204
26	·45379	45553	45728	45902	46077	46251	46426	46600	46775	46949
27	·47124	47298	47473	47647	47822	47997	48171	48346	48520	48695
28	·48869	49044	49218	49393	49567	49742	49916	50091	50265	50440
29	·50615	50789	50964	51138	51313	51487	51662	51836	52011	52185
30	·52360	52534	52709	52883	53058	53233	53407	53582	53756	53931
31	·54105	54280	54454	54629	54803	54978	55152	55327	55501	55676
32	·55851	56025	56200	56374	56549	56723	56898	57072	57247	57421
33	·57596	57770	57945	58119	58294	58469	58643	58818	58992	59167
34	·59341	59516	59690	59865	60039	60214	60388	60563	60737	60912
35	·61087	61261	61436	61610	61785	61959	62134	62308	62483	62657
36	·62832	63006	63181	63355	63530	63705	63879	64054	64228	64403
37	·64577	64752	64926	65101	65275	65450	65624	65799	65973	66148
38	·66323	66497	66672	66846	67021	67195	67370	67544	67719	67893
39	·68068	68242	68417	68591	68766	68941	69115	69290	69464	69639
40	·69813	69988	70162	70337	70511	70686	70860	71035	71209	71384
41	·71558	71733	71908	72082	72257	72431	72606	72780	72955	73129
42	·73304	73478	73653	73827	74002	74176	74351	74526	74700	74875
43	·75049	75224	75398	75573	75747	75922	76096	76271	76445	76620
44	·76794	76969	77144	77318	77493	77667	77842	78016	78191	78365

Differences

1′	2′	3′	4′	5′
29	58	87	116	145

$90° = 1{\cdot}57080^{c}$ $270° = 4{\cdot}71239^{c}$

$180° = 3{\cdot}14159^{c}$ $360° = 6{\cdot}28319^{c}$

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′
45°	·78540	78714	78889	79063	79238	79412	79587	79762	79936	80111
46	·80285	80460	80634	80809	80983	81158	81332	81507	81681	81856
47	·82030	82205	82380	82554	82729	82903	83078	83252	83427	83601
48	·83776	83950	84125	84299	84474	84648	84823	84998	85172	85347
49	·85521	85696	85870	86045	86219	86394	86568	86743	86917	87092
50	·87266	87441	87616	87790	87965	88139	88314	88488	88663	88837
51	·89012	89186	89361	89535	89710	89884	90059	90234	90408	90583
52	·90757	90932	91106	91281	91455	91630	91804	91979	92153	92328
53	·92502	92677	92852	93026	93201	93375	93550	93724	93899	94073
54	·94248	94422	94597	94771	94946	95120	95295	95470	95644	95819
55	·95993	96168	96342	96517	96691	96866	97040	97215	97389	97564
56	·97738	97913	98088	98262	98437	98611	98786	98960	99135	99309
57	·99484	99658	99833	*00007*	*00182*	*00356*	*00531*	*00706*	*00880*	*01055*
58	1·01229	01404	01578	01753	01927	02102	02276	02451	02625	02800
59	1·02974	03149	03323	03498	03673	03847	04022	04196	04371	04545
60	1·04720	04894	05069	05243	05418	05592	05767	05941	06116	06291
61	1·06465	06640	06814	06989	07163	07338	07512	07687	07861	08036
62	1·08210	08385	08559	08734	08909	09083	09258	09432	09607	09781
63	1·09956	10130	10305	10479	10654	10828	11003	11177	11352	11527
64	1·11701	11876	12050	12225	12399	12574	12748	12923	13097	13272
65	1·13446	13621	13795	13970	14145	14319	14494	14668	14843	15017
66	1·15192	15366	15541	15715	15890	16064	16239	16413	16588	16763
67	1·16937	17112	17286	17461	17635	17810	17984	18159	18333	18508
68	1·18682	18857	19031	19206	19381	19555	19730	19904	20079	20253
69	1·20428	20602	20777	20951	21126	21300	21475	21649	21824	21999
70	1·22173	22348	22522	22697	22871	23046	23220	23395	23569	23744
71	1·23918	24093	24267	24442	24617	24791	24966	25140	25315	25489
72	1·25664	25838	26013	26187	26362	26536	26711	26885	27060	27235
73	1·27409	27584	27758	27933	28107	28282	28456	28631	28805	28980
74	1·29154	29329	29503	29678	29852	30027	30202	30376	30551	30725
75	1·30900	31074	31249	31423	31598	31772	31947	32121	32296	32470
76	1·32645	32820	32994	33169	33343	33518	33692	33867	34041	34216
77	1·34390	34565	34739	34914	35088	35263	35438	35612	35787	35961
78	1·36136	36310	36485	36659	36834	37008	37183	37357	37532	37706
79	1·37881	38056	38230	38405	38579	38754	38928	39103	39277	39452
80	1·39626	39801	39975	40150	40324	40499	40674	40848	41023	41197
81	1·41372	41546	41721	41895	42070	42244	42419	42593	42768	42942
82	1·43117	43292	43466	43641	43815	43990	44164	44339	44513	44688
83	1·44862	45037	45211	45386	45560	45735	45910	46084	46259	46433
84	1·46608	46782	46957	47131	47306	47480	47655	47829	48004	48178
85	1·48353	48528	48702	48877	49051	49226	49400	49575	49749	49924
86	1·50098	50273	50447	50622	50796	50971	51146	51320	51495	51669
87	1·51844	52018	52193	52367	52542	52716	52891	53065	53240	53414
88	1·53589	53764	53938	54113	54287	54462	54636	54811	54985	55160
89	1·55334	55509	55683	55858	56032	56207	56382	56556	56731	56905

Differences

1′	2′	3′	4′	5′
29	58	87	116	145

$1^c = 57° 17' 45''$ $\quad$ $3^c = 171° 53' 14''$

$2^c = 114° 35' 30''$ $\quad$ $4^c = 229° 10' 59''$

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	1000	1020	1040	1061	1082	1103	1124	1145	1166	1188	2	4	6	8	10	13	15	17	19
11	1210	1232	1254	1277	1300	1323	1346	1369	1392	1416	2	5	7	9	11	14	16	18	21
12	1440	1464	1488	1513	1538	1563	1588	1613	1638	1664	2	5	7	10	12	15	17	20	22
13	1690	1716	1742	1769	1796	1823	1850	1877	1904	1932	3	5	8	11	13	16	19	22	24
14	1960	1988	2016	2045	2074	2103	2132	2161	2190	2220	3	6	9	12	14	17	20	23	26
15	2250	2280	2310	2341	2372	2403	2434	2465	2496	2528	3	6	9	12	15	19	22	25	28
16	2560	2592	2624	2657	2690	2723	2756	2789	2822	2856	3	7	10	13	16	20	23	26	30
17	2890	2924	2958	2993	3028	3063	3098	3133	3168	3204	3	7	10	14	17	21	24	28	31
18	3240	3276	3312	3349	3386	3423	3460	3497	3534	3572	4	7	11	15	18	22	26	30	33
19	3610	3648	3686	3725	3764	3803	3842	3881	3920	3960	4	8	12	16	19	23	27	31	35
20	4000	4040	4080	4121	4162	4203	4244	4285	4326	4368	4	8	12	16	20	25	29	33	37
21	4410	4452	4494	4537	4580	4623	4666	4709	4752	4796	4	9	13	17	21	26	30	34	39
22	4840	4884	4928	4973	5018	5063	5108	5153	5198	5244	4	9	13	18	22	27	31	36	40
23	5290	5336	5382	5429	5476	5523	5570	5617	5664	5712	5	9	14	19	23	28	33	38	42
24	5760	5808	5856	5905	5954	6003	6052	6101	6150	6200	5	10	15	20	24	29	34	39	44
25	6250	6300	6350	6401	6452	6503	6554	6605	6656	6708	5	10	15	20	25	31	36	41	46
26	6760	6812	6864	6917	6970	7023	7076	7129	7182	7236	5	11	16	21	26	32	37	42	48
27	7290	7344	7398	7453	7508	7563	7618	7673	7728	7784	5	11	16	22	27	33	38	44	49
28	7840	7896	7952	8009	8066	8123	8180	8237	8294	8352	6	11	17	23	28	34	40	46	51
29	8410	8468	8526	8585	8644	8703	8762	8821	8880	8940	6	12	18	24	29	35	41	47	53
30	9000	9060	9120	9181	9242	9303	9364	9425	9486	9548	6	12	18	24	30	37	43	49	55
31	9610	9672	9734	9797	9860	9923	9986				6	13	19	25	31	38	44	50	57
31								1005	1011	1018	1	1	2	3	3	4	5	5	6
32	1024	1030	1037	1043	1050	1056	1063	1069	1076	1082	1	1	2	3	3	4	5	5	6
33	1089	1096	1102	1109	1116	1122	1129	1136	1142	1149	1	1	2	3	3	4	5	5	6
34	1156	1163	1170	1176	1183	1190	1197	1204	1211	1218	1	1	2	3	3	4	5	6	6
35	1225	1232	1239	1246	1253	1260	1267	1274	1282	1289	1	1	2	3	4	4	5	6	6
36	1296	1303	1310	1318	1325	1332	1340	1347	1354	1362	1	1	2	3	4	4	5	6	7
37	1369	1376	1384	1391	1399	1406	1414	1421	1429	1436	1	2	2	3	4	5	5	6	7
38	1444	1452	1459	1467	1475	1482	1490	1498	1505	1513	1	2	2	3	4	5	5	6	7
39	1521	1529	1537	1544	1552	1560	1568	1576	1584	1592	1	2	2	3	4	5	6	6	7
40	1600	1608	1616	1624	1632	1640	1648	1656	1665	1673	1	2	2	3	4	5	6	6	7
41	1681	1689	1697	1706	1714	1722	1731	1739	1747	1756	1	2	2	3	4	5	6	7	7
42	1764	1772	1781	1789	1798	1806	1815	1823	1832	1840	1	2	3	3	4	5	6	7	8
43	1849	1858	1866	1875	1884	1892	1901	1910	1918	1927	1	2	3	3	4	5	6	7	8
44	1936	1945	1954	1962	1971	1980	1989	1998	2007	2016	1	2	3	4	4	5	6	7	8
45	2025	2034	2043	2052	2061	2070	2079	2088	2098	2107	1	2	3	4	5	5	6	7	8
46	2116	2125	2134	2144	2153	2162	2172	2181	2190	2200	1	2	3	4	5	6	7	7	8
47	2209	2218	2228	2237	2247	2256	2266	2275	2285	2294	1	2	3	4	5	6	7	8	9
48	2304	2314	2323	2333	2343	2352	2362	2372	2381	2391	1	2	3	4	5	6	7	8	9
49	2401	2411	2421	2430	2440	2450	2460	2470	2480	2490	1	2	3	4	5	6	7	8	9
50	2500	2510	2520	2530	2540	2550	2560	2570	2581	2591	1	2	3	4	5	6	7	8	9
51	2601	2611	2621	2632	2642	2652	2663	2673	2683	2694	1	2	3	4	5	6	7	8	9
52	2704	2714	2725	2735	2746	2756	2767	2777	2788	2798	1	2	3	4	5	6	7	8	9
53	2809	2820	2830	2841	2852	2862	2873	2884	2894	2905	1	2	3	4	5	6	7	9	10
54	2916	2927	2938	2948	2959	2970	2981	2992	3003	3014	1	2	3	4	5	7	8	9	10

Find the position of the decimal point by inspection.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
55	3025	3036	3047	3058	3069	3080	3091	3102	3114	3125	1	2	3	4	6	7	8	9	10
56	3136	3147	3158	3170	3181	3192	3204	3215	3226	3238	1	2	3	5	6	7	8	9	10
57	3249	3260	3272	3283	3295	3306	3318	3329	3341	3352	1	2	3	5	6	7	8	9	10
58	3364	3376	3387	3399	3411	3422	3434	3446	3457	3469	1	2	4	5	6	7	8	9	11
59	3481	3493	3505	3516	3528	3540	3552	3564	3576	3588	1	2	4	5	6	7	8	10	11
60	3600	3612	3624	3636	3648	3660	3672	3684	3697	3709	1	2	4	5	6	7	8	10	11
61	3721	3733	3745	3758	3770	3782	3795	3807	3819	3832	1	2	4	5	6	7	9	10	11
62	3844	3856	3869	3881	3894	3906	3919	3931	3944	3956	1	3	4	5	6	7	9	10	11
63	3969	3982	3994	4007	4020	4032	4045	4058	4070	4083	1	3	4	5	6	8	9	10	11
64	4096	4109	4122	4134	4147	4160	4173	4186	4199	4212	1	3	4	5	6	8	9	10	12
65	4225	4238	4251	4264	4277	4290	4303	4316	4330	4343	1	3	4	5	7	8	9	10	12
66	4356	4369	4382	4396	4409	4422	4436	4449	4462	4476	1	3	4	5	7	8	9	11	12
67	4489	4502	4516	4529	4543	4556	4570	4583	4597	4610	1	3	4	5	7	8	9	11	12
68	4624	4638	4651	4665	4679	4692	4706	4720	4733	4747	1	3	4	5	7	8	10	11	12
69	4761	4775	4789	4802	4816	4830	4844	4858	4872	4886	1	3	4	6	7	8	10	11	13
70	4900	4914	4928	4942	4956	4970	4984	4998	5013	5027	1	3	4	6	7	8	10	11	13
71	5041	5055	5069	5084	5098	5112	5127	5141	5155	5170	1	3	4	6	7	9	10	11	13
72	5184	5198	5213	5227	5242	5256	5271	5285	5300	5314	1	3	4	6	7	9	10	12	13
73	5329	5344	5358	5373	5388	5402	5417	5432	5446	5461	1	3	4	6	7	9	10	12	13
74	5476	5491	5506	5520	5535	5550	5565	5580	5595	5610	1	3	4	6	7	9	10	12	13
75	5625	5640	5655	5670	5685	5700	5715	5730	5746	5761	2	3	5	6	8	9	11	12	14
76	5776	5791	5806	5822	5837	5852	5868	5883	5898	5914	2	3	5	6	8	9	11	12	14
77	5929	5944	5960	5975	5991	6006	6022	6037	6053	6068	2	3	5	6	8	9	11	12	14
78	6084	6100	6115	6131	6147	6162	6178	6194	6209	6225	2	3	5	6	8	9	11	13	14
79	6241	6257	6273	6288	6304	6320	6336	6352	6368	6384	2	3	5	6	8	10	11	13	14
80	6400	6416	6432	6448	6464	6480	6496	6512	6529	6545	2	3	5	6	8	10	11	13	14
81	6561	6577	6593	6610	6626	6642	6659	6675	6691	6708	2	3	5	7	8	10	11	13	15
82	6724	6740	6757	6773	6790	6806	6823	6839	6856	6872	2	3	5	7	8	10	12	13	15
83	6889	6906	6922	6939	6956	6972	6989	7006	7022	7039	2	3	5	7	8	10	12	13	15
84	7056	7073	7090	7106	7123	7140	7157	7174	7191	7208	2	3	5	7	8	10	12	14	15
85	7225	7242	7259	7276	7293	7310	7327	7344	7362	7379	2	3	5	7	9	10	12	14	15
86	7396	7413	7430	7448	7465	7482	7500	7517	7534	7552	2	3	5	7	9	10	12	14	16
87	7569	7586	7604	7621	7639	7656	7674	7691	7709	7726	2	4	5	7	9	10	12	14	16
88	7744	7762	7779	7797	7815	7832	7850	7868	7885	7903	2	4	5	7	9	11	12	14	16
89	7921	7939	7957	7974	7992	8010	8028	8046	8064	8082	2	4	5	7	9	11	13	14	16
90	8100	8118	8136	8154	8172	8190	8208	8226	8245	8263	2	4	5	7	9	11	13	14	16
91	8281	8299	8317	8336	8354	8372	8391	8409	8427	8446	2	4	5	7	9	11	13	15	16
92	8464	8482	8501	8519	8538	8556	8575	8593	8612	8630	2	4	6	7	9	11	13	15	17
93	8649	8668	8686	8705	8724	8742	8761	8780	8798	8817	2	4	6	7	9	11	13	15	17
94	8836	8855	8874	8892	8911	8930	8949	8968	8987	9006	2	4	6	8	9	11	13	15	17
95	9025	9044	9063	9082	9101	9120	9139	9158	9178	9197	2	4	6	8	10	11	13	15	17
96	9216	9235	9254	9274	9293	9312	9332	9351	9370	9390	2	4	6	8	10	12	14	15	17
97	9409	9428	9448	9467	9487	9506	9526	9545	9565	9584	2	4	6	8	10	12	14	16	18
98	9604	9624	9643	9663	9683	9702	9722	9742	9761	9781	2	4	6	8	10	12	14	16	18
99	9801	9821	9841	9860	9880	9900	9920	9940	9960	9980	2	4	6	8	10	12	14	16	18

Find the position of the decimal point by inspection.

ANSWERS

Note. Four-figure tables have been used for working out the Answers given below. Angles are given to the nearest minute, but in many cases there are variations of one or two minutes, depending on the use made of the Tables.

PART I.

EXERCISE I. a. (p. 5.)

1. 270 ; 45 ; 60. **2.** 70° ; 155° ; 125° 40′ ; 17° 10′ ; 42° 35′.

3. 50° ; 63° 30′ ; 26° 10′ ; 87° ; 31° 50′.

4. 90° ; 58° ; 120° ; 110° ; 30°. **5.** 70° ; 170° ; 200° ; 310°.

6. N. 50° E. ; S. 30° W. ; N. 2° W. ; S. 70° E.

7. S. 10° W. ; N. 14° W. ; 195° ; 130°.

8. ∠ of depr. 18° 45′. **9.** 15° 27′.

10. 28° 22′ 19″ ; 25° 42′ 51″ ; 2° 46′ 40″ ; 4° 12′ 24″.

11. 25·590 ; 108·289. **12.** 15′. **13.** 30′.

14. 84° 35′ ; 11° 30′. **15.** N. 25° W.

EXERCISE I. b. (p. 7.)

1. 1·87, 1·40 in. ; 0·47, 0·47. **2.** 5·72 cm., 1·71 in. ; 0·57, 0·57.

3. 3 cm., 1·8 in. ; 6 cm., 10 in. ; 0·6, 0·6.

4. 11·2 cm., 8 in. ; 3·75 cm., 7·5 in. ; 1·6, 1·6. **5.** 72 ft.

6. 860,000 mi. ; 230,000 mi. **7.** 5 ft.

8. 10 × 6 cm. ; 5 × 3 in., yes. **9.** No. **10.** 4·8 in.

EXERCISE I. c. (p. 13.)

1. 0·3640, 0·8391, 1, 1·1918, 1·7321, 3·7321.

2. 14° 2′, 41° 59′, 58°, 66° 30′. **3.** 1·8807, 0·6346, 0·2820, 2·0248.

4. 23°, 71°, 15° 24′, 71° 36′, 88°, 14° 14′, 50° 14′, 41° 44′.

5. 38° 40′, 56° 19′, 62° 59′, 36° 52′.
6. 26° 34′, 10° 18′; 42° 1′, 47° 59′, 35°; 29° 3′.
7. 3·70, 6·88, 6·54, 20·5. 8. 5·69; 196, 83·9; 6·15.
9. $\sqrt{3}$, $\frac{1}{3}\sqrt{3}$; 1·7321, 0·5774. 10. 1. 11. 2·26 cm., 11·1 cm., 1·80 in.
12. 199 ft. 13. 818 ft. 14. 34° 42′. 15. 2·02 in.
16. 13·0 ft 17. S. 75° 58′ E., S. 63° 26′ E. 18. 56° 19′.
19. 21° 48′. 20. 48° 49′.
21. 63° 26′, 26° 34′, 90°; 38° 40′, 51° 20′, 90°. 22. 7·98 ft.
23. 67° 23′, 112° 37′. 24. 15·4 cm. 25. 21·7 sq. in.
26. 48′. 27. 2·36 cm. 28. 30° 58′.
29. 12° 55′. 30. 52·9 ft. 31. 10·7 in.
32. 3° 37′. 33. 1·73 in. 34. 5·02 cm.

EXERCISE I. d. (p. 18.)

1. 385 ft. 2. 15·0 ft. 3. 25° 38′. 4. 77° 19′.
5. 47° 31′. 6. 1, 0. 7. 246 ft. 8. 14·0 ft.
9. 3·92 in. 10. 8·83, 4·40 cm.; 52° 55′. 11. 1·27 ft.
12. 68° 59′. 13. 31° 54′. 14. 35° 16′. 15. 18° 26′.
16. 3·79 cm. 17. 1·99 cm. 18. 15·5, 12·6, 15·5 ft.
19. 37° 28′. 20. 2·80 in.

EXERCISE II. a. (p. 26.)

1. 0·423, 0·906, 0·574, 0·819, 0·906, 0·423; 1, 0, 0, 1.
2. 0·2924, 0·6820, 0·8988, 0·9994, 0·3987, 0·4003, 0·3990, 0·4000, 0·6205, 0·9002, 0·7551, 0·9965.
3. 0·9703, 0·8829, 0·5592, 0·0175, 0·3971, 0·3955, 0·3963, 0·3958, 0·4592(1), 0·9959, 0·9969, 0·5995.
4. $\sin\theta$, $\cos\phi = 0{\cdot}6$, $\cos\theta$, $\sin\phi = 0{\cdot}8$; $\sin\theta$, $\cos\phi = \frac{5}{13}$, $\cos\theta$, $\sin\phi = \frac{12}{13}$; $\sin\alpha$, $\cos\theta = 0{\cdot}8$, $\cos\alpha$, $\sin\theta = 0{\cdot}6$; $\sin\beta$, $\cos\phi = 0{\cdot}6$, $\cos\beta$, $\sin\phi = 0{\cdot}8$; $\sin\theta$, $\cos\phi = 0{\cdot}28$, $\cos\theta$, $\sin\phi = 0{\cdot}96$.
5. sin B, cos C; sin R, cos P; sin E, cos F; tan Z; tan B; sin X, cos Z; tan F; sin C, cos B; tan P; sin F, cos E; sin Z, cos X.
6. $x = 4{\cdot}54$, $y = 8{\cdot}91$; $a = 2{\cdot}65$, $b = 1{\cdot}41$; $p = 3{\cdot}11$, $q = 2{\cdot}52$; $e = 84{\cdot}8$, $f = 53{\cdot}0$.
7. 2·77, 0·740; 94·6, 3·60, 5·43, 1·65, 1·83, 1·35.
8. 0·8, 0·6, $\frac{4}{3}$; 0·96, 0·28, $\frac{24}{7}$; $\frac{\sqrt{3}}{2}$, $\frac{1}{2}$, $\sqrt{3}$; 0·8, 0·6, $1\frac{1}{3}$.
9. 92·7 ft. 10. 1120, 1660 yd. 11. 353 ft.
12. 16·0 sq. in. 13. 8480. 14. 42°, 42° 30′, 68° 13′, 74° 39′.
15. 0·9683. 16. 6·41 cm. 17. 23·9 cm. 18. 6·30, 6·25 cm.

19. 0·997 in. 20. 6·95 ft. 21. 5·88 cm. 22. 1·55 ft.
23. 11·9, 1·83 cm. 24. 10·7 ft. 25. 478 yd.

EXERCISE II. b. (p. 31.)

1. $23\frac{1}{2}^\circ$, $44\frac{1}{2}^\circ$, $48\frac{1}{2}^\circ$, 67°. 2. 72°, 51°, 36°, $24\frac{1}{2}^\circ$.
3. 23°, 74°, 50° 18′, 26° 42′, 26° 48′, 26° 44′, 26° 46′, 13° 34′, 18° 39′, 74° 44′ (45′).
4. 56°, 38°, 29° 24′, 79° 36′, 79° 42′, 79° 40′, 79° 37′, 40° 40′, 30° 20′, 10° 28′.
5. 31° 46′, 71° 56′, 32° 15′, 58° 39′, 39′, 71° 37′.
6. 36° 52′, 53° 8′; 22° 37′, 67° 23′; $\alpha, \phi = 53° 8′$, $\beta, \theta = 36° 52′$; 16° 16′, 73° 44′.
7. 14° 29′. 8. 5° 44′. 9. 53° 8′, 73° 44′, 60°, 53° 8′.
10. 36° 52′, 143° 8′.
11. 19° 28′, 18° 26′; 5° 44′, 5° 43′; 1° 54′, 1° 54′; 34′, 34′; along slope.
12. 9° 35′. 13. 33° 33′. 14. 46° 39′. 15. 18°, 0·39 in.
16. 89° 36′. 17. 24° 2′. 18. 36° 26′. 19. 11° 29′.
20. 24° 37′. 21. 31° 48′, 106° 48′. 22. 38° 56′.
23. 112° 53′. 24. 23° 35′. 25. 69° 42′. 26. 208° 58′.
27. 51° 24′, 51° 50′. 28. 24° 37′. 29. 11·6 cm. 30. 46° 22′, 88° 51′.
31. 3·69. 32. 2·41, 3·19, 4·37.

EXERCISE II. c. (p. 35.)

1. 2410 mi. 2. 15,000 mi. 3. 52·9, 140 yd. 4. 5·47, 11·9 ft.
5. 46·9 yd. 6. 2·40, 1·01 mi. 7. 67° 8′. 8. N. 53° 34′ E.
9. 113° 35′. 10. 1·31 in. 11. 158, $(174t - 16t^2)$ ft.; 10·9 sec.
12. 7·95 ft. 13. 7·24, 2·76 in. 14. 64° 14′. 15. 41° 24′.
16. $\frac{AN}{AC}$, $\frac{AC}{AB}$. 17. 88° 50′. 18. 4490 yd. 19. C = 90°.
20. 69. 22. $x + y = 90$, 30. 23. 35, 53.
24. 15° 22′. 25. 41° 49′. 26. 32° 30′. 27. 12° 50′.

EXERCISE III. a. (p. 41.)

1. 1·5243, 1·5062, 1·5052, 1·0335, 1·1326, 1·1357, 1·1364, 3·0281, 1·0355, 0·9896, 0·9885, 0·1198.
2. 56° 30′, 56° 26′, 24° 34′, 44° 24′, 44° 50′, 63° 40′, 56° 12′, 56° 9′, 32° 8′.
3. $\theta\ \frac{5}{3}, \frac{5}{4}, \frac{4}{3}, \phi\ \frac{5}{4}, \frac{5}{3}, \frac{3}{4}$; $\theta\ \frac{13}{5}, \frac{13}{12}, \frac{12}{5}, \phi\ \frac{13}{12}, \frac{13}{5}, \frac{5}{12}$; $\alpha, \phi\ \frac{5}{4}, \frac{5}{3}, \frac{3}{4}, \beta, \theta\ \frac{5}{3}, \frac{5}{4}, \frac{4}{3}$; $\theta\ \frac{25}{7}, \frac{25}{24}, \frac{24}{7}, \phi\ \frac{25}{24}, \frac{25}{7}, \frac{7}{24}$.
4. cosec C, sec B; cosec P, sec R; tan F, cot E; cosec X, sec Z; tan B, cot C; cosec R, sec P; cosec F, sec L; tan X, cot Z; sin C, cos B; tan R, cot P; cosec E, sec F; cosec Z, sec X; sin P, cos R; cosec B, sec C; sin R, cos P.

5. $\frac{CB}{AC}, \frac{PQ}{QR}, \frac{EF}{EG}, \frac{YZ}{YX}, \frac{AB}{AC}, \frac{PR}{QR}, \frac{EG}{GF}, \frac{XZ}{YZ}, \frac{CB}{CA}, \frac{PQ}{PR}$, BC, PR, GE.

6. 2·9238, 0·6157, 1·2349, 0·8746, 1, 1, 1, 1.

7. 36° 2′, 29° 36′, 60° 7′, 38° 21′.

8. 9·15, 6·90 ; 5·82, 4·99 ; 8·31, 6·63.

9. 0·6626 ; 0·2946 ; 1·6552.

10. 28°, 76°, 38° 35′, 74° 18′, 70° 13′, 18° 50′.

11. 1·1100.

12. $\frac{x}{p}, \frac{p+q}{x}$; $\frac{q}{h}, \frac{y}{x}$; $\frac{x}{p}$; $\frac{p}{h}=\frac{h}{q}$; $\frac{y}{q}=\frac{p+q}{y}=\frac{x}{h}$; $\frac{x}{h}=\frac{p+q}{y}=\frac{y}{q}$.

13. 492 ft. 14. 5·22 cm. 15. 358 ft. 16. 88·9 ft.

17. 4·19 ft. 18. 18·6 ft. 19. 19·1 min. 20. 5·28 in.

21. 35·4 ft. 22. 1460 yd. 23. 0·32 in. 24. 9·45 cm.

25. 5·23, 17·2 in. 26. 11·3, 6·35, 11·7 cm. 27. 2·34 in.

28. 6·61. 29. 4·68 cm. 30. 6·27, 10·7 cm.

EXERCISE III. b. (p. 45.)

1. 15·0 ft. 3. $h(\cot\phi - \cot\theta)$ ft. 5. 4·88 cm., 11·3 cm.

6. $a(\sec\theta - \cos\theta)$. 7. $h \operatorname{cosec}\theta + \frac{1}{2}c\sec\theta$.

8. $a\sec^3\theta$, $a\sec\theta(\sec^2\theta - 1) = a\sec\theta\tan^2\theta$.

9. 5·08, 6·50 in. ; 63° 30′.

10. $c=6·73$, $b=5·38(5)$, $a=4·45$, A = 41° 12′, B = 52° 54′, C = 85° 54′.

11. 107·(5) yd. 12. 40·5 in. 13. 39·7 in. 14. $2d \operatorname{cosec}^3\theta$.

15. $p \operatorname{cosec}\theta - x\cot\theta$. 16. 10·9, 4·54 cm. 17. 0·3201.

18. C = 90°. 19. A = 90°. 20. 63. 21. 21.

22. $x+y=90$, 18. 23. 18. 24. 35.

REVISION PAPERS. R. 1-6. (p. 48.)

R. 1. 2. 21·2 cm. 3. 24·2 sq. in. 4. 2·34, 3·81 ft. 5. 10° 19′.

R. 2. 2. 168 ft. 3. 65° 23′, 65° 23′, 49° 14′.
4. 117 ft./sec. 5. 16° 22′.

R. 3. 1. 15° 31′. 2. 302 yd. 3. 82°.
4. 2·1525, 1·2152, 0·3007, 1. 5. 4·88.

R. 4. 1. $b=13·6$, $a=9·16$. 2. 3·19. 3. 35° 47′, 8° 24′.
4. 47° 10′. 5. 3860 yd.

R. 5. 1. 35° 33′, 14·0 cm. 2. 2° 24′. 3. 9° 36′, 9° 28′.
4. 1560 yd. 5. 2·85 cm.

R. 6. 1. 90°. 2. 56° 26′ or 123° 34′. 3. 10·5, 9·74 cm.
4. 60·7(5) cm. 5. 19° 27′.

EXERCISE IV. a. (p. 53.)

1. $\sqrt{2}$. 2. $\dfrac{2}{\sqrt{3}}$. 3. 1. 4. 2. 5. $\frac{1}{3}\sqrt{3}$.
6. $\sqrt{2}$. 7. 2. 8. $\sqrt{3}$. 9. $\sqrt{3}$. 10. $\frac{1}{3}\sqrt{3}$.
11. 1. 12. 1. 13. 45° or vertical. 14. 30°.
15. 60°. 16. 6, $3\sqrt{3}=5\cdot20$ in. 17. 27·7 sq. cm.
18. 17·3, 14·1, 10 ft. 20. Is trebled. 21. In 20 sec. more.
22. $\sqrt{3}-1$, $\frac{1}{2}(\sqrt{3}-1)$, 15°. 23. $\dfrac{\sqrt{3}+1}{2\sqrt{2}}$.
24. $\sin\theta$, $\cos\theta$, $\tan\theta$; 1, 0, ∞ ; 0, 1, 0.

EXERCISE IV. b. (p. 56.)

1. 1·1918. 2. 0·2924. 3. 0·3640. 4. 0·4226. 5. 2·3559.
6. 1·0576. 7. 1·1918. 8. 0·2679. 9. 0·75, 1·25. 10. $1\frac{2}{3}$, 2·6.
11. $\frac{1}{4}\sqrt{7}$, $1\frac{1}{3}$. 12. 2, $\frac{1}{2}\sqrt{5}$. 13. 1·96, 1. 14. $\sin\theta$, $\sec\theta$, $\cos\theta$.
15. 1, $\sin^2\theta$, $\cot\theta$. 20. $\sqrt{(p^2-1)}$, $\dfrac{1}{p}\sqrt{(p^2-1)}$. 22. $\cot\theta$.
23. (ii), (v), (vii), (x), (xii). 24. 0, ∞. 25. 0. 0.
26. 1. 27. 1. 29. 98, 45°.
30. $\dfrac{\sqrt{3}+1}{2\sqrt{2}}$; $\dfrac{\sqrt{3}+1}{2\sqrt{2}}$, $\dfrac{\sqrt{3}-1}{2\sqrt{2}}$, $2+\sqrt{3}$.

EXERCISE IV. c. (p. 58.)

1. $z\cos\theta$. 2. $x\operatorname{cosec}\theta$. 3. $y\operatorname{cosec}\phi$. 4. $y\cot\phi$.
5. $\tan^{-1}\left(\dfrac{y}{x}\right)$. 6. $\sin^{-1}\left(\dfrac{x}{z}\right)$. 7. $z\sin\theta$. 8. $x\sec\phi$.
9. $\tan^{-1}\left(\dfrac{x}{y}\right)$. 10. $\cos^{-1}\left(\dfrac{x}{z}\right)$. 11. $x\cot\theta$. 12. $y\sec\theta$.
13. PQ cosec R. 14. GF cot E. 15. YZ cosec X. 16. $\cos^{-1}\left(\dfrac{QR}{PR}\right)$.
17. $\tan^{-1}\left(\dfrac{YZ}{XY}\right)$. 18. PQ cot R. 19. XZ cos X. 20. EF sin E.
21. 6°. 22. 328 ft. 23. 17° 28′. 24. 25° 40′.
25. 38° 56′. 26. 61° 3′. 27. 22° 1′, 38° 41′. 28. 4° 39′.
29. 50° 29′. 30. 32° 37′. 31. 5·64 ft. 32. 2·85 in.
33. 4° 55′. 34. 7·28 chn. 35. 51° 42′.
36. 8° 5′, 5° 25′, 25 ft. 37. 34′. 38. 6·20, 8·14 cm.
39. 341 yd. 40. 2·57 mi.

EXERCISE IV. d. (p. 62.)

1. 38° 56′. 2. 1·15 in. 3. 9·52 cm.
4. 77° 19′, 32° 1′ incr. 5. 56° 26′. 6. 4·75 cm.

7. 4·26 ft. 8. 87°. 9. 6·25, 4·92(5) in.
10. 30·0, 49·9 ft. 11. $\sec^2\theta$.
12. 3·08 ft.; 3·99, 3·14, 4·95 ft.; 4·53, 2·72, 1·87 ft.
13. $(b-c)\sin\theta + d\cos\theta = a$. 15. $a\tan\theta - b$; 56° 19′.
17. 18° 26′, 36° 52′. 19. $d\sin^2\theta\cos\theta$, $d(1-\sin^2\theta\cos^2\theta)$.
20. 90° or 36° 52′.

EXERCISE V. a. (p. 70.)

1. 22° 1′. 2. 13° 21′. 3. 67° 23′. 4. 17° 55′.
5. 27° 56′. 6. 36° 52′. 7. 75° 58′. 8. 45°.
9. 36° 52′. 10. 26° 34′. 11. 45° 14′. 12. 96° 40′.
13. 39° 31′. 14. 40° 36′. 15. 49° 24′. 16. 100° 57′.
17. 66° 6′. 18. 45°. 19. 81° 12′. 20. 40° 36′.
21. 10·06, 115° 3′. 22. 28° 28′. 23. N. 59° 2′, E. or W.
24. 10° 4′, 16° 28′. 25. 9·60 ft. 26. 16·6 ft.
27. 29° 34′. 28. 87° 43′, 56° 19′. 29. 141 ft.
30. 1920 ft.
31. OA = AC = 2, OC = $\sqrt{6}$, 52° 15′, 52° 15′, 75° 30′; $\triangle$OAC $\equiv$ $\triangle$OBC; $\sqrt{3}$, $\frac{1}{2}\sqrt{15} = 1·94$; 63° 25′. 32. 71·4 ft.
33. 36° 52′, 40° 54′, 141° 48′. 34. 2·65 in., 43° 11′.
35. 37° 23′, 31° 43′.

EXERCISE V. b. (p. 74.)

1. 11° 50′, 18° 59′, 49° 11′. 2. 81° 59′. 3. 18° 36′.
4. 28° 23′. 5. 63° 58′. 6. 23° 59′. 7. 58° 23′.
8. 35° 50′. 9. 54° 28′, 28° 2′. 10. $\frac{1}{13}$. 11. 27° 51′.
12. 89° 33′. 13. 40° 49′. 14. 10° 2′. 15. N. 51° 5′ E. or W.
16. S. 67° 4′ E. 17. 35° 16′, 109° 26′. 18. 15° 35′ or 143° 1′.
19. 26° 34′. 21. 93° 11′.

EXERCISE VI. a. (p. 80.)

1. 0·9998. 2. 1·0002. 3. 0·0175. 4. 57·30.
5. 57·29. 6. 0·0175. 7. 0·0087. 8. 114·6.
9. 1·000. 10. 1·000. 11. 0·0087. 12. 114·6.
13. $x > 89$. 14. $x > 89·4$. 15. $x > 87·7$. 16. $x < 1$.
17. $x < 0·6$. 18. $x < 2·3$. 19. $x < 0·6$. 20. $x < 0·6$.
21. $x > 89·4$. 22. $x > 89$. 23. $x > 89·4$. 24. $x < 0·6$.
25. $x = 0$; ∞, 1, ∞; $x = 90$; 1, ∞, 0. 26. cot 0° is ∞, cot 90° = 0.

EXERCISE VI. b. (p. 81.)

1. Each is image of other. 2. 0·31, 0·59, 0·95, 0·95, 0·81, 0·31.
3. 34°, 76°, 56°, 26°, 45°. 4. 53°. 5. 48°.

EXERCISE VI. c. (p. 83.)

1. Graph of cot $x°$. 2. Graph of tan $x°$. 3. tan 90°, cot 0° are ∞.
4. 0·44, 1·48, 3·49, 3·49, 0·70, 0·25.
5. 13°, 63°, 71½°, 14°, 45°. 6. 55°. 7. 48°.

EXERCISE VI. d. (p. 85.)

1. 3·33(5), 3·53, 3·54, 3·34. 2. 42°, 70°; 34°, 78½°; 39°, 73°; 31°.
3. 2·28, 1·44, 1·10, 1·10, 1·24, 3·86; 15°, 73°, 65°, 33½°, 45°.
6. 5, 82½°. 7. 57°. 8. 12½°. 9. 1·24, 36°.
10. 37½°. 11. 20½°. 12. 51¼.
13. 0·00075, 0·00075; 0·0007; a straight line.
14. Straight line, curve; 0·0012, 0·0012; 0·35, 0·61. 15. 18.
16. 0 or 30. 17. 15. 18. 0, 65½.
19. 32¾ or 72. 20. 30. 21. 21 ft.
22. 43. 23. 51. 24. 41¾.

REVISION PAPERS. R. 7-18. (p. 88.)

R. 7. 1. 136 ft. 2. 96° 40′. 3. ½, 3.
4. 10, 30. 5. 47° 10′.

R. 8. 1. $\frac{1}{2}\sqrt{5}=1·118$. 2. 9·37 in. 3. 10·2, 8·12 ft.
4. 90°, 14° 29′. 5. 15° 47′.

R. 9. 1. 2·97, 8·93 mi. 2. 2, $\frac{8}{3}$. 3. 2·80 ft.
4. 11° 32′, 53° 8′. 5. 71° 34′.

R. 10. 1. 10·4, 5·98 in. 2. $x > 63·4$, $x > 60$, $x > 45$.
3. 198 ft. 5. 17° 58′.

R. 11. 1. $\frac{2}{7}\sqrt{6}=0·700$. 2. 7·13 ft. 3. 12·1 cm.
4. 75° 31′, 28° 58′; 63° 26′, 26° 34′.
5. 4·23, 13·3 cm.; 17° 23′, 50° 7′.

R. 12. 1. 0·086(5) amp. 2. 24·7, 13·9 in. 3. 35° 7′.
4. 3·16. 5. 55° 30′; S. 39° 6′ W.

R. 13. 1. 13·0, 32° 28′. 2. 66° 25′. 3. 1·91 cm.
4. 24·8. 5. 18° 26′.

R. 14. 1. 0·186, 1. 2. 28·9 cm. 3. 14·8 in.
4. 54¾; 67 or 40½. 5. 11° 28′.

R. 15. 1. $\frac{\sqrt{(b^2-1)}}{b}$, $\sqrt{(b^2-1)}$. **2.** 32° 53′. **3.** 1·45 ft.
4. $\sqrt{(p^2+q^2)}$; 68° 12′. **5.** 14° 44′.

R. 16. 1. $\frac{m^2-1}{2m}$, $\frac{2m}{m^2+1}$. **2.** 70·4, 20·7 mi.; 16° 24′.
3. 39° 56′. **4.** 80°. **5.** 5·7 in., 25° 6′.

R. 17. **1.** 30°, 30°, 60°. **2.** 1·30 ft. **3.** 9° 36′.
4. $19\frac{1}{2}$ or $13\frac{3}{4}$. **5.** 14° 3′.

R. 18. **1.** $x^2+y^2=25$. **3.** 20·7 per cent. **4.** 114 ft.
5. 6·16 cm.

ANSWERS

PART II

EXERCISE VII. a. (p. 98.)

1. (0·4, 0·5) ; (−0·7, 0·3) ; 0·4, 0·5, 0·7, 0·3 in.
2. (−0·3, −0·6) ; (0·6, −0·3) ; 0·3, 0·6, 0·6, 0·3 in.
3. No ; one of four positions. **4.** G, H. **5.** C, D ; E, F.
6. x is −, y is +; x is +, y is −; x is +, y is +; x is −, y is −.

EXERCISE VII. b. (p. 103.)

1. −0·8, −0·6, $\frac{4}{3}$. **2.** 0·8, −0·6, $-\frac{4}{3}$. **3.** −0·8, 0·6, $-\frac{4}{3}$.
4. 0·6, −0·8, $-\frac{3}{4}$. **5.** −0·6, 0·8, $-\frac{3}{4}$. **6.** −0·6, −0·8, $\frac{3}{4}$.
7. 0·8, −0·6, $-\frac{4}{3}$. **8.** −0·6, 0·8, $-\frac{3}{4}$. **9.** $-\frac{\sqrt{21}}{5}$, −0·4, $\frac{\sqrt{21}}{2}$.
10. 0·906 ; −0·423 ; −0·766 ; 0·839 ; −0·819 ; 0·574 ; −0·985 ; −0·839.
11. $90 < \theta < 180$. **12.** $270 < \theta < 360$. **13.** $90 < \theta < 180$.
14. $180 < \theta < 270$. **15.** $270 < \theta < 360$. **16.** $270 < \theta < 360$.
17. −cos 20°. **18.** sin 10°. **19.** −sin 20°. **20.** cos 80°.
21. −sin 10°. **22.** −cos 15°. **23.** −sin 80°. **24.** −cos 70°.
25. −tan 35°. **26.** sin 85°. **27.** tan 50°. **28.** −tan 35°.
29. 113° 35′, 246° 25′. **30.** 228° 36′, 311° 24′. **31.** 153° 26′, 333° 26′.
32. 30° 58′, 210° 58′. **33.** 0·3420. **34.** −0·9659.
35. −1·7321. **36.** −0·9063. **37.** 0·5774. **38.** −0·6820.
39. 0·7314. **40.** −0·5736. **41.** 0·8323. **42.** −0·5543.
43. 0·7378. **44.** 0·6237.

EXERCISE VII. c. (p. 105.)

1. 0·89, −0·89, −0·89, 0·89 ; 0·89, −0·89, −0·89.
2. 107·5, 252·5 ; 197·5, 342·5. **3.** 53, 127 ; 37, 323.

4. $180 < x < 360$; $90 < x < 270$.

5. $23{\cdot}5 < x < 156{\cdot}5$; $203{\cdot}5 < x < 336{\cdot}5$.

6. $0 < x < 66{\cdot}5$ or $293{\cdot}5 < x < 360$; $113{\cdot}5 < x < 246{\cdot}5$.

7. 45, 225. **10.** 60°, 300°. **11.** 20°, 160°. **12.** 58°, 238°.

13. 230°, 310°. **14.** 117°, 243°. **15.** 158°, 338°. **16.** −2·9238.

17. 1·5557. **18.** −3·7321. **19.** −1·3054. **20.** 0·0875.

21. 1·0154. **22.** −1·5557. **23.** −0·8391.

24. 63° 26′, 243° 26′. **25.** 23° 35′, 156° 25′. **26.** 114° 38′, 245° 22′.

27. 204° 37′, 335° 23′. **28.** 63° 26′, 116° 34′, 243° 26′, 296° 34′.

29. 54° 44′, 125° 16′, 234° 44′, 305° 16′.

32. 143° 8′ ; 306° 52′ ; 323° 8′. **33.** $\operatorname{cosec}\theta$. **34.** $-\sec\theta$.

35. $-\cot\theta$. **36.** $\sec\theta$. **37.** $-\operatorname{cosec}\theta$. **38.** $\cot\theta$.

39. $\sin A$; $-\cos A$.

EXERCISE VII. d. (p. 109.)

1. 2·5, 2·5, −2·5, −2·5 ft. **2.** 3·83 ft., 13 sec.

3. $5\cos(10t^\circ)$ feet ; 4·33, −4·33, −4·33, 4·33 ft.

4. 6·64, 29·36 sec. ; 11·36, 24·64 sec.

5. 7·05, −11·4, 11·4, −7·05 ft.

6. 5·7 a.m., 5.37 p.m. ; 11.22 a.m., 11.52 p.m.

7. 2·5, −2·5, −5, −2·5, 2·5, 5 ft ; 10 ft. ; 12 sec.

8. $15+4\cos\theta$, $4\sin\theta$ mi. ; 18·8, 1·37 ; 11·2, 1·37 ; 19, 0 ; 11, 0 ; 19, 0 ; 11·2, −1·37 ; 18·8, −1·37.

9. Yes ; 0. **10.** Yes ; $2l$, l, 0. **11.** Yes. **12.** Yes.

14. $p\sin\alpha+q\sin\beta+r\sin\gamma$; −1·355, 0·756 ; 1·162, −0·123 ; −1·45(5), −0·387.

15. 90°, 340°, 60°, 160°.

16. $\cot\theta=\frac{1}{2}(\cot B-\cot C)$; 73° 35′, 49° 42′, 106° 25′

EXERCISE VIII. (p. 115.)

1. 0·8192, $\bar{1}{\cdot}9134$, $\bar{1}{\cdot}9134$; 0·7944, $\bar{1}{\cdot}9000$, $\bar{1}{\cdot}9000$; 1·1303, 0·0532, 0·0532 ; 0·0822, $\bar{2}{\cdot}9149$, $\bar{2}{\cdot}9150$; 4·4277, 0·6462, 0·6462 ; 2·5096, 0·3995, 0·3995.

2. $\bar{1}{\cdot}4805$; $\bar{1}{\cdot}6490$; $\bar{1}{\cdot}6600$; $\bar{1}{\cdot}4925$; 1·0106 ; 1·0106.

3. 53° or 127° ; 28° 36′ or 331° 24′ ; 71° 36′ or 251° 36′ ; 51° 30′ or 231° 30′ ; 28° 48′ or 331° 12′ ; 5° 42′ or 174° 18′ ; 20° 44′ or 159° 16′ ; 38° 28′ or 218° 28′ ; 67° 33′ or 292° 27′ ; 70° 28′ or 250° 28′ ; 15° 39′ or 164° 21′ ; 75° 28′ or 284° 32′.

4. 1·22. **5.** 0·294. **6.** 0·633. **7.** 0·519.

8. 2·84. **9.** 1·78. **10.** 87·9. **11.** 0·193(5).

12. 0·424. **13.** 1·08. **14.** 1·11. **15.** 139.

16. 38° 24′. **17.** 56° 37′. **18.** 65° 36′. **19.** 18° 28′.
20. 54° 39′. **21.** 27° 35′. **22.** 22° 13′. **23.** 29° 4′.
24. 66° 5′. **25.** 76° 19′. **26.** 1·55. **27.** 57·1.
28. 86·7. **29.** 27° 15′. **30.** 91° 6′, 51° 54′. **31.** 56° 7′.
32. 36° 14′. **33.** 5·40. **34.** 91° 8′. **35.** 24° 16′.

EXERCISE IX. a. (p. 117.)

1. One. **2.** None. **3.** One. **4.** One.
5. One. **6.** Any number. **7.** Two. **8.** One.
9. None. **10.** One. **11.** None. **12.** One.
13. None. **14.** Two. **15.** $A+B+C=180°$. **16.** $b+c>a$, etc.

EXERCISE IX. b. (p. 122.)

1. 9·40. **2.** 36·3. **3.** 6·40. **4.** 7·97.
5. 6·96. **6.** 7·23. **7.** 35° 43′. **8.** 47° 29′.
9. 41° 48′. **10.** 41° 23′. **11.** 61° 6′. **12.** 29° 39′.
13. Two. **14.** One. **15.** None. **16.** One. **17.** Two.
18. One. **19.** One. **20.** One. **21.** None. **22.** Two.
23. $10>b>7{\cdot}88$; $b=7{\cdot}88$ or $b>10$; $b<7{\cdot}88$. **24.** B.
25. $c>10$; No. **26.** 66° 5′ or 113° 55′. **27.** None.
28. 38° 45′ or 5° 15′. **29.** None. **30.** 90°.
31. 104° 37′. **32.** None. **33.** 19° 47′.
34. 17° 39′ or 57° 27′. **35.** 63° 50′, 7·71, 5·84.
36. 77° 27′, 51° 18′, 6·24 or 102° 33′, 26° 12′, 3·53.
37. 117° 55′, 2·01(5), 2·23. **38.** 15° 50′, 36° 50′, 2·90.
39. 80° 6′, 45° 19′, 4·52 or 99° 54′, 25° 31′, 2·74.
40. 35° 50′, 4·02, 6·92. **41.** 56° 11′, 60° 19′, 7·99.
42. 46° 47′, 43° 13′, 6·79. **43.** 28° 3′, 61° 57′, 10·3(5).
44. 31° 30′, 16°, 271.
45. 87° 53′, 52° 47′, 1010 or 13° 27′, 127° 13′, 235.

EXERCISE IX. c. (p. 128.)

[*Note.* The answers in this and the following Exercises are given to a higher degree of accuracy than will always be attained if the (four-figure) table of squares is employed.]

1. 1·19. **2.** 7·07(5). **3.** 6·34(5). **4.** 2·97(5).
5. 46° 34′. **6.** 43° 32′. **7.** 109° 28′. **8.** 120°.
9. 13° 9′, 145° 21′, 3·22. **10.** 33° 48′, 44° 4′, 7·03.
11. 20° 42′, 143° 54′, 4·51. **12.** 14° 57′, 25° 30′, 7·54.

13. 93° 50′, 56° 15′, 29° 55′.
14. 100° 17′, 43° 32′, 36° 11′.
15. 97° 54′, 52° 25′, 29° 41′.
16. 35° 6′, 109° 48′, 35° 6′.
17. 83° 55′, 58° 45′, 5·26.
18. 15° 58′, 22° 37′, 9·45(5).
19. 38° 11′, 47° 59′, 93° 50′.
20. 105° 3′, 33° 17′, 41° 42′.
21. 25° 17′, 35° 28′, 656.
22. 144° 25′, 13° 5′, 5·31.
23. 74° 13′, 58° 33′, 47° 14′.
24. 18° 12′, 121° 12′, 40° 36′.
25. 52° 26′, 75° 8′, 52° 26′.
26. 34° 15′, 34° 15′, 11·2.

EXERCISE IX. d. (p. 129.)

1. 40° 12′, 189, 128.
2. 86° 22′, 54° 52′, 38° 46′.
3. 78° 43′, 53° 56′, 92·3 or 101° 17′, 31° 22′, 59·4.
4. 25° 25′, 34° 50′, 13·8.
5. 118° 11′, 4·06, 3·33.
6. 107° 44′, 32° 4′, 40° 12′.
7. 23° 44′, 32° 1′, 2·89.
8. 10° 30′, 15° 12′, 40·2.
9. 53°, 89° 40′, 23·4 or 127°, 15° 40′, 6·32.
10. 17° 55′, 99·7, 176.
11. 25° 56′, 36° 31′, 117° 33′.
12. 20° 25′, 38·3, 58·4.
13. 68° 15′, 60° 15′, 163 or 8° 45′, 119° 45′, 26·6.
14. 46°, 71° 25′, 33·8.
15. 59° 30′, 61°, 37·5.

EXERCISE IX. e. (p. 129.)

1. 54, 66 mi.
2. 1010 yd.
3. 13·8 mi.
4. 8·67(5), 8·67 mi.
5. 24 mi.
6. 111 ft.
7. 0·13(5) mi.
8. 206 yd.
9. 23° 33′.
10. 133 yd.
11. 12 yd.
12. 200 yd.
13. 2420 yd.
14. 5° 8′.
15. 4470 yd.
16. 12·7 sea mi.
17. 3650 yd., $335\frac{3}{4}$°.
18. 10200 yd.
20. 6520 ft.
21. 19 m.p.h.
22. 500 yd.
23. 6·17 ft.
24. 24° 9′; 18·5 ft.
25. 110° 47′.

EXERCISE IX. f. (p. 132.)

1. 47·6 ft.
2. 828 ft.
3. 4·03, 2·97 ft.
4. 62° 43′.
5. 18° 12′, 41° 24′.
6. 7·11 in.
7. 6·32(5).
8. 4·34.
9. 0.
10. 18·8, 1·2 in.; 0·7, 19·3 in.
11. 16·4 in., 86° 8′.
12. 22·4 in.
13. 49·1 in.
14. 18·6 in.
15. 21·6(5) ft.
16. 15·3(5) ft.
17. 6·32 ft.
18. 11·9 ft.
19. N. 67° 9′ W.
20. $1-2\sin^2\theta$.
21. 3·88 in.
22. 1·16.
24. 57° 58′ or 35° 20′.
25. Right, 0·49 mi.; left, 0·79(5) mi.
26. 100° 57′.

27. 22·3, 20·4 in.
28. 4100 ft.
29. 12, 4, 10·6, 10·6, ; 60°.
30. 2·10 in.

REVISION PAPERS. R. 19-26. (p. 137.)

R. 19. 1. 68° 12′, 111° 48′. 3. 3·45. 4. 5° 8′. 5. 8·44(5) cm.

R. 20. 1. 11° 19′. 2. 3·46, 3·46, 0, −3·46, −3·46 in. 3. 55° 25′. 4. 83° 38′; 3·92 in. 5. 38° 13′.

R. 21. 2. 3·76, 2·05 in. 3. 100 ft.; 17° or 73°. 4. 56° 15′, 56° 15′, 67° 30′. 5. 5·4(5), 24·9 ft.

R. 22. 1. 42·3 sq. cm. 2. 0·67, 5, 9·33, 9·33, 5, 0·67 in. 3. 1·56 sq. ft. 4. 1 yd. 5. 4° 39′.

R. 23. 1. −0·9428. 2. 8·54. 3. 26° 34′. 4. 83° 20′. 5. 3·05(5) in.

R. 24. 1. 7·00 in. 2. 11° 15′. 3. 6·23(5) in. 4. 7·92 in. 5. 2·37 in.

R. 25. 1. 1·16 in. 2. $xy+7=x+y$. 3. 4·48 cm. 4. ±0·9948; ±0·9137. 5. 113° 58′.

R. 26. 1. $\frac{1}{m}$ or m. 2. 30·0. 3. $6\cos\theta$, $4\sin\theta$. 5. 1·17 in.

EXERCISE X. a. (p. 146.)

1. 3·14. 2. 3·12. 3. 3·17.
4. $4r^2$, $2r^2$ sq. cm.; $4>\pi>2$.
6. 44 cm., 154 sq. cm.; 14·8 cm., 17·3(5) sq. cm.
7. 3·61, 4·48(5) in. 8. 480. 9. 18·5 m. per sec.
10. 75·8 sq. cm 11. 77 in. 12. 6·98 cm.
13. 27·9 sq. cm. 14. 57° 18′. 15. 0·35 in.
17. 12·6 in. 18. 1·8 ft. 19. 31·6 ft.
20. 7·75 cm. 21. 1·8(5), 1·8 sq. in. 22. 298 sq. cm.
23. 37·3 sq. ft., 23·8(5) ft. 24. 302 cu. in., 251 sq. in.
25. 1·78 in. 26. 12. 27. 1885 sq. ft. 28. 25 sq. cm.
29. 13·3 sq. in. 30. 3·0 sq. in., 13·5 in. 31. 23·4 ft.
32. 162 cm., 12·2 (5). 33. 0·825 cm. 35. 37·7 ft.; 5000 ft.
36. 22·2 in. 37. 84° 33′.

EXERCISE X. b. (p. 152.)

1. 2° 30′; 10 min. 2. 829, 2420 mi. 3. 21600.
4. 42 mi. 5. 4 hr. 42 min. 6. 12·4 ft.
7. 39,800 stad.; 4,570 mi. 8. 70° 32′. 9. 48° 11′.
10. 171 mi. (statute). 11. 851 mi.; 12° 17′; 848·(5) mi.

EXERCISE X. c. (p. 156.)

1. 52·4 cu. cm., 65·8 sq. cm. **2.** 410·(5) cu. ft., 234 sq. ft.
3. 16·4 cu. in. **4.** 277°.
5. 25° 41′. **6.** 14·1 cu. in., 28·3 sq. in.
7. 197,000,000 sq. mi.; 35,200,000 sq. mi. **8.** 14·9 in.
9. 6·77 in. **10.** $2d$ in. **11.** 8,150,000 sq. mi.
12. 134 cu. ft., 151 sq. ft. **13.** 25·7 cu. in.
14. 102° 38′; 151 sq. in.; 402 cu. in.; 204 cu. in.; 198 cu. in.
15. 203 cu. cm.; 28° 4′; 126 sq. cm. **16.** 53° 8′.
17. 0·79 cm. **18.** $y \sin 2\theta \tan (45° - \theta) = \dfrac{y \sin 2\theta \cos 2\theta}{1 + \sin 2\theta}$ cm.
19. $\theta = 360 \sin \phi$. **20.** 1·4 in. **21.** 34,700 sq. ft.
22. 4010. **23.** 453 sq. cm. **25.** 36° 52′.

EXERCISE XI. a. (p. 163.)

1. 57° 18′, 171° 53′, 28° 39′, 63° 1·(5)′, 143° 14′, 45° 50′, 4° 0·6′, 14′.
2. 90°, 135°, 36°, 150°, 15°, 67° 30′, 270°, 315°, 80°, 105°.
3. $\frac{3\pi}{2}, \frac{\pi}{3}, \frac{5\pi}{6}, \frac{3\pi}{4}, \frac{5\pi}{12}, \frac{\pi}{5}, \frac{3\pi}{5}, \frac{\pi}{8}, \frac{7\pi}{4}, \frac{7\pi}{6}$.
4. 0·2967, 0·6807, 1·0122, 1·5010, 0·0102, 0·0157, 0·3124, 0·8133(5), 1·2988, 2·2294.
5. 75·8, 24·9, 126·8. **6.** 20, 13·4, 5·5, 6·56, 27·5 cm.
7. 0·75, 1·25, 3, 0·59. **8.** 0·6458, 32·3 sq. cm.
9. 1, -1, -1, -1, $\frac{1}{2}$, $\frac{1}{2}$, 1, $\frac{1}{2}\sqrt{3}$, $\sqrt{3}$, 1, $-\frac{1}{2}$, $\frac{1}{2}\sqrt{2}$.
10. $\sin\theta$, $\sin\theta$, $-\cot\theta$, $-\cos\theta$, $-\sin\theta$, $-\tan\theta$, $-\cos\theta$, $\sin\theta$, $-\cot\theta$, $\tan\theta$, $-\sin\theta$, $-\cos\theta$.
11. 1·5, 0·75, $\pi - 0·75 = 2·39$ rad. **12.** 1·2, 68° 46′, 5·65 in.
13. 1·2, 11·3 cm. **14.** 5 ft./sec. **15.** $\frac{2\pi}{3}$.
16. 24 sq. cm. **17.** $\frac{5\pi}{12}$. **18.** $\frac{\pi}{3}$. **19.** 10.
20. 60·3. **21.** $r\omega$ ft./sec. **22.** 1·5, 2·5 cm.,; 4 sq. cm.
23. $29\frac{1}{3}$. **24.** 2·39 : 1. **25.** 0·8415, 1·0768, 1·4019.
26. 0·122, 0·0785, 0·0116, 0·000174(5), 0·105, 0·0175, 0·0436(5), 0·0873.
28. $2r \sin\left(\frac{\theta}{2}\right)$. **29.** $a\left[1 - \cos\left(\frac{b}{a}\right)^c\right]$. **31.** 57° 21′.
32. 11·9 cm. **34.** $a + (l - m)\sin\left(\frac{m}{a}\right) - a\cos\left(\frac{m}{a}\right)$. **35.** $\frac{s}{a}$ rad.

EXERCISE XI. b. (p. 170.)

1. 57 in. **2.** 1° 35′. **3.** 2160 mi. **4.** 1° 9′.
5. $2{\cdot}6 \times 10^{13}$ mi. **6.** 92,300,000 mi. **8.** 11 ft. **9.** 25′.
10. 22·6 in. **11.** 0·9848, 0·999894. **12.** 2460 ft.
13. 94,400,000 mi. **14.** 15 mi. **15.** 7·52′, 15·0′.
16. 67 ft. **17.** 24·4 mi. **18.** 19 min.
19. 3180 ft. **20.** 0·4 rad.; $\frac{\pi}{3}$ sec. **21.** 3, $\frac{5\pi}{2}$.
22. 3900 mi. **23.** 0·054(5) in. **24.** 0·00079 in.
25. 0·97. **26.** 0·50756.

EXERCISE XI. c. (p. 174.)

1. 0·64 or 2·5. **2.** 0·96. **3.** 0·36 or 2·78. **4.** 0·52.
5. $\pm 0{\cdot}79(5)$. **6.** $\pm 1{\cdot}32$. **7.** 1·9. **8.** 2·28.
9. 2·68. **10.** 2·75. **11.** 0·51. **12.** 0·83.
13. 0·74. **14.** 0·98. **15.** 0·52.
16. 0·74, 1·03, 1·98, 1·28, 1·28. **18.** 4·49. **19.** 127°.
20. 117°. **21.** 86°. **22.** 132°. **23.** 0·58.
24. 1·27. **25.** 36° = 0·63 rad. **26.** 10, 9 ft.; 3 ft./sec.
27. $2a, \lambda; v$.

EXERCISE XII. a. (p. 180.)

1. 6·95. **2.** 9·92. **3.** 10·0. **4.** 25·3.
5. 18·7. **6.** 5440. **7.** 43·0. **8.** 18·9.
9. 23·8(5). **10.** 24° 41′ or 155° 19′.
11. 117° 17′, 36° 20′, 143° 40′, 62° 43′; 12·4 sq. in. **12.** 39° 48′.
13. 3·12 in. **14.** 195 ac. **15.** 7·15 cm. **16.** 2·75 sq. ft.
17. 39° 8′. **18.** 10·5. **19.** 68·4 sq. cm. **20.** 12 per cent.
21. 5·10, 4·13 in. **22.** 28° 26′. **23.** 209 sq. ft.
24. 0·829 sq. ft. **25.** 26° 47′.
26. $l(\cos\theta - \sin\theta\tan\theta) = l\sec\theta\cos 2\theta$;
$l^2\sin\theta(\cos\theta - \sin\theta\tan\theta) = l^2\tan\theta\cos 2\theta$.
29. $\dfrac{2bc\cos\theta}{b+c}$.

EXERCISE XII. b. (p. 183.)

1. 512 cu. cm., 576 sq. cm. **2.** 39·5 cu. in.
3. 320 cu. cm., 45 sq. cm., 185 cu. cm. **4.** 380 cu. in.
5. 341 sq. in. **6.** 272 ton. **7.** 6875 cu. ft.
8. 7·54 cu. in. **9.** 128 cu. cm. **10.** $\tan\beta = \sqrt{2}\,.\tan\alpha$.

EXERCISE XII. c. (p. 186.)

1. 1·96 in. **2.** 3·63, 3·63 in. **3.** 3·57, 1·63 in.
4. 2, 3, 6 in. **5.** 3·31, 1·62. **6.** 1·52, 0·674.
7. 25° 22′. **8.** 7·56 mi. **9.** 11·2 cm.
10. 5·01(5). **11.** 3, 2 in.; 12, 2 cm. **12.** 2, 9 in.; 2, 32 cm.
13. $\sqrt{[abc(a+b+c)]}$ sq. in.; $\sqrt{\left[\dfrac{abc}{a+b+c}\right]}$; $\sqrt{\left[\dfrac{bc(a+b+c)}{a}\right]}$ in.; etc.
14. 7·42 in.

REVISION PAPERS. R. 27-34. (p. 187.)

R. 27. **1.** 25·7 in., 10·7 ft. **3.** 2·06(5), 15·2.
4. 4′ 46″. **5.** 30·4.
R. 28. **1.** 44·4 cm., 114 sq. cm. **2.** 47 ft.
4. Main roads, 4·9 min. **5.** 98·4, 100·8 per cent.
R. 29. **1.** 0·3 per cent. **2.** 48·6, 55·4 chn. **3.** 1·01 in.
4. 460 sq. in. **5.** 90 sq. in.; 59° 29′.
R. 30. **1.** 10·7 ft. **2.** 4·49, 1·98 ft. **3.** 52° 27′ N.
4. 3·51(5) ft., 362 cu. ft. **5.** $a\sqrt{2}$.
R. 31. **2.** 9·5(5)′. **3.** 36° 52′, 11° 32′.
4. 75·7 sq. cm., 38·3(5) cu. cm. **5.** 6·64 cm.
R. 32. **1.** 26° 23′. **2.** 6·43 cm., 28·1 sq. cm.
3. 13° 28′ or 86° 32′. **4.** 136 yd.
5. 96° 14′, 18·3 sq. in., 29° 12′.
R. 33. **1.** 29° 27′. **3.** 15° 1′. **4.** 96 ft. **5.** 5·36 cm.
R. 34. **1.** 25·5 cm., 4·9 cm. **2.** 11·8 cm. **3.** 17° 43′.
4. $h \cot\theta \operatorname{cosec}\theta$.

$\bar{4}\ \bar{2} + \bar{2}.1364$

$\bar{4}. + 2.1364$

$\bar{1}. 5341$

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